CS 530: Geometric and Probabilistic Methods in Computer Science
Homework 5 (Fall '13)

1. A bivariate Gaussian r.v., $\mathbf{x} = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^{\mathrm{T}}$, has the following p.d.f.:

$$f(x_0, x_1) = \frac{(a^2 - b^2)^{\frac{1}{2}}}{2\pi} \exp\left(-\frac{1}{2} \left[a x_0 x_0 + 2b x_0 x_1 + a x_1 x_1 \right] \right).$$

   (a) Give the matrix $\mathbf{W}$ which will decorrelate the components of $\mathbf{x}$.

   (b) Let $\mathbf{u} = \mathbf{W}\mathbf{x}$. Give an expression for $g(u_0, u_1)$, the p.d.f. for the bivariate Gaussian random variable, $\mathbf{u} = \begin{bmatrix} u_0 & u_1 \end{bmatrix}^{\mathrm{T}}$.

2. Let $\mathbf{x}_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}_3 = \begin{bmatrix} -1 & 0 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}_4 = \begin{bmatrix} 0 & -1 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}_5 = \begin{bmatrix} 0 & -2 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{x}_6 = \begin{bmatrix} -2 & 0 \end{bmatrix}^{\mathrm{T}}$, be samples of a two-dimensional vector random variable. Do the following:

   (a) Compute the covariance matrix for the vector random variable.

   (b) Compute the eigenvectors of the covariance matrix.

   (c) Compute the KL-transform of $\mathbf{x}_1$.

3. Assume that the red, green, and blue components of the cactus image (cactus.ppm) found on the class homepage are samples of a 3×1 vector random variable. Use MATLAB to do the following:

   (a) Compute the mean vector and subtract it from each of the pixels to produce a color image with zero mean.

   (b) Compute the 3×3 covariance matrix for the zero mean image.

   (c) Compute the eigenvectors and eigenvalues of the covariance matrix.

   (d) Transform each pixel in the cactus image using the matrix of eigenvectors of the covariance matrix (i.e., KL transform).

   (e) Display the transformed image. If you have access to a color printer, provide hardcopy of the tranformed image. Otherwise provide hardcopy of the red, green, and blue components of the transformed image.

   (f) Show that the covariance matrix of the transformed image is diagonal. Show that the variances are equal to the eigenvalues of the covariance matrix of the untransformed image.

4. Prove the following:

$$\frac{\left[\frac{P(x+\Delta x,t)-P(x,t)}{\Delta x} - \frac{P(x,t)-P(x-\Delta x,t)}{\Delta x}\right]}{\Delta x} = \left.\frac{\partial^2 P}{\partial x^2}\right|_{x,t} + \mathrm{O}[(\Delta x^2)].$$

Hint. Replace $P(x+\Delta x,t)$, and $P(x-\Delta x,t)$ with their Taylor series.

5. The partial differential equation (PDE) which governs the motion of waves propagating with velocity c in a two-dimensional medium is:

$$\frac{\partial^2 P}{\partial t^2} = c^2\left[\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right].$$

We will consider the evolution of functions of the torus. More precisely, we assume that P is periodic in x and y with periods, X and Y, i.e., $P(x,y) = P(x+jX, y+kY)$ for all integers, j and k.

- Using finite difference approximations for $\partial^2 P/\partial t^2$, $\partial^2 P/\partial x^2$, and $\partial^2 P/\partial y^2$, derive an update equation for numerical solution of the two-dimensional wave equation, i.e., an equation for $P^t_{x,y}$ in terms of $P^{t-1}_{x,y}$, $P^{t-2}_{x,y}$, $P^{t-1}_{x-1,y}$, etc.
- Implement and test your numerical scheme in MATLAB. Compute solutions of the wave equation for functions of a torus of size 64 m $\times$ 64 m. You should assume that $\Delta x = \Delta y$ is 1 m, Δt is 0.005 s, and the conduction velocity c is 100 m/s. The initial condition $P^0_{x,y} = P^1_{x,y}$ is a Gaussian of standard deviation, 3 m. For reasons of symmetry, the location of the mean is not important. Render your solutions as images (256 grey values) and provide hardcopy for $t = 0.1$ s, 0.2 s, ..., 2.0 s.
- **Hint.** The trickiest thing about this problem is properly computing the update on the boundaries of the periodic domain. This is relatively painless (and quite fast) if you recognize that the principal term in the update equation is actually a convolution which can be computed using the 2D FFT. For example, the second partial derivative in the x direction of a discrete $N \times N$ image $\mathbf{F}$ can be estimated at every location by convolving $\mathbf{F}$ with the $N \times N$ mask $\mathbf{G}$:

$$\left.\frac{\partial^2 F}{\partial x^2}\right|_{x,y} \approx \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} F(i,j)\cdot G(x - i \bmod N, y - j \bmod N)$$

where

$$G = \begin{bmatrix} -2 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Consequently, $\frac{\partial^2 F}{\partial x^2} \approx \mathbf{F} * \mathbf{G} = \mathbf{W}\left(\widehat{\mathbf{F}} \cdot \widehat{\mathbf{G}}\right)\mathbf{W}$ where $\widehat{\mathbf{F}} = \mathbf{W}^* \mathbf{F} \mathbf{W}^*$ and $\widehat{\mathbf{G}} = \mathbf{W}^* \mathbf{G} \mathbf{W}^*$.

If you choose not to use the FFT, you can still make your code fast by using vector operations instead of *for* loops whenever possible.

- **Optional stuff.** You can make your solutions colorful by rendering them as color images. One way to do this is to let P^t be red, P^{t-1} be green, and P^{t-2} be blue. You can animate your solutions using the Linux *animate* utility. Experiment with other values for c. At what velocity does the solution become numerically unstable? What does numerical instability look like?

6. There is a link called *Color Matching Curves* on the class webpage to a Matlab file which defines seven vectors. The vectors R, G, and B represent the values of the spectral sensitivity functions of the red, green, and blue cones of the human visual system at 69 equally spaced wavelengths between 390 nm and 730 nm. The vectors X, Y, and Z represent the values of the CIE 1931 standard color matching functions, and the fourth vector, P, represents the reflectance distribution of a petunia (all at the same 69 wavelengths).

   (a) Compute the tristimulus values (*i.e.,* X, Y and Z) for the petunia using the CIE 1931 color matching functions.

   (b) Compute the chromaticities (*i.e.,* x, y and z) for the petunia, from the tristimulus values.

   (c) Based on the CIE 1931 chromaticity diagram, what color is the petunia?