Tiling groups for Wang tiles *

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Abstract

We apply tiling groups and height functions to tilings of regions in the plane by Wang tiles, which are squares with colored boundaries where the colors of shared edges must match. We define a set of tiles as unambiguous if it contains all tiles equivalent to the identity in its tiling group. For all but one set of unambiguous tiles with two colors, we give efficient algorithms that tell whether a given region with colored boundary is tileable, show how to sample random tilings, and how to calculate the number of local moves or "flips" required to transform one tiling into another. We also analyze the lattice structure of the set of tilings, and study several examples with three and four colors as well.

1 Introduction

Tilings of the plane with Wang tiles [1, 8] have been studied in computer science since the famous result of Berger [4] that the problem of whether we can tile the infinite plane using a given set of Wang tiles is undecidable. This paper focuses on tilings of a given finite region with colored boundary. This is a well-known NP-complete problem [15, 10] and we intend to tackle the subproblem in which the number of colors is fixed. Our approach is algebraic: we use the *tiling groups* of Conway and Lagarias [7], and *height functions*, introduced by Thurston [26] and independently in the statistical physics literature (see [6] for a review). These ideas were used and generalized by Kenyon and Kenyon [13], Rémila [23, 25], Propp [21], and others, for the problem of tiling planar regions with different types of polyominoes or simple polygons. Our work is, to our knowledge, the first time Wang tiles have been addressed with these techniques.

We define a set of tiles as unambiguous if a certain algebraic condition is both necessary and sufficient for single tiles. For all but one unambiguous set of two-color tiles, we give a polynomial-time algorithm to tell whether a given region with given colors on its boundary is tileable. We also study the structure of the set of tilings under local "flips" that change the color of a few interior edges, and show that this is either a distributive lattice or a hypercube. In particular, this graph is connected, i.e. any tiling can be turned into any other with a series of flips, and we give a formula for the minimum number of flips necessary to do so. In several cases, these tilings turn out to be equivalent to familiar systems with height functions, such as domino tilings and Eulerian orientations. We can then apply the techniques of Luby, Randall and Sinclair [16] and Propp and Wilson [22] to sample random tilings in polynomial time.

We finish by carefully studying a set of tiles with three colors, and by noting that some sets of three- and four-color tiles possess two- and three-dimensional height functions. We also note that several sets of tiles are isomorphic in the sense that there is a natural bijection between pairs of tilings and boundary conditions, even though their tilings groups are not isomorphic.

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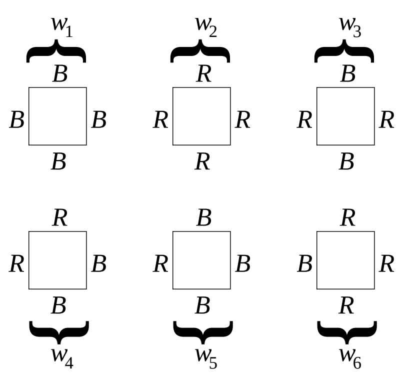
2 The tiling group

Let Λ be the square lattice of the Euclidean plane $\mathbb{R}^2$. A (finite) region P of Λ is a (finite) union of closed square cells of Λ. A region P is said to be a polygon if its interior and its complement $\mathbb{R}^2 \setminus P$ are both connected.

A Wang tile is a square of side one with colored edges. An assignment of Wang tiles to the cells of a polygon P corresponds to a tiling if tiles on neighboring cells have the same color along their common edge. Throughout the paper, the "boundary conditions" of a tiling will include not just the shape of the region, but the colors on its boundary.

Let S be a set of Wang tiles constructed with two colors, *B*lue and *R*ed. Let P be a polygon with colors B and R on the edges of its boundary. We study the problem of finding a tiling of P using the tiles of S in such a way that the colors of the boundary condition are satisfied.

Let $W = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ be the set of all Wang tiles with two colors. i.e.



Note that we allow these tiles to be rotated. A subset $\{w_{x_1}, \cdots, w_{x_n}\} \subseteq W$ will be denoted by $W_{x_1 \cdots x_n}$.

To solve the tiling problem for particular subsets of W, we start by introducing an orientation of the edges of Λ. First, we will assume that the squares of Λ are colored black and white like a checkerboard. We orient the edges of Λ so that they go clockwise and counterclockwise around black and white squares respectively, so that an ant going along an edge will have a white square on its left and a black square on its right.

Whenever we have a tiling T of a polygon P with tiles in W, the colors of the edges of P will be either B or R. Let us write a symbol b whenever we move along a blue edge with the orientation, and b^{-1} when we move against it. Similarly, we write r and r^{-1} for moving along a red edge. To every tiled polygon P with colored edges we can associate a contour word $w \in \{b, r, b^{-1}, r^{-1}\}^*$ starting from any external vertex and following a path around the boundary of P. Let $S \subseteq W$ and let v be the set of contour words of the tiles in S. Then the tiling group $G_S = \langle b, r \mid v \rangle$ is the free group modulo the relations $w = e$ for each contour word $w \in v$. This can also be written as a factor $G_S = \langle b, r \rangle / N_S$ where N_S is the normal subgroup generated by the contour words in v and their conjugates.

Note that $uv = e$ if and only if $vu = e$, and that for square two-color tiles, every mirror image is also a rotation. Thus it doesn't matter where we start on a tile, or in which direction we go around it, to define its contour word; we obtain the same tiling group G_S. On the other hand, for three or more colors, we would have to explicitly allow reflections as well as rotations.

Now that we have defined the tiling group for tiles with colored edges, we make several observations. First, any tiling T of a polygon P with a set of tiles S corresponds to a *tiling function* $f_T : V \to G_S$, where V is the set of vertices in P or on its boundary. We do this by first fixing f on a particular vertex, say $f_T(x_0) = e$, where x_0 is the leftmost vertex of the bottom of P. We then define f inductively as follows: If we have already assigned an element $x \in G_S$ to a vertex v and if the oriented edge $(v, u) \in P$ is colored with B (resp. R), then set $f_T(u) = xb$ (resp. $f_T(u) = xr$). Similarly, if $(u, v) \in P$ is colored B (resp. R) then set $f_T(u) = xb^{-1}$ (resp. xr^{-1}). Thus moving along the arrows, or against them, changes the value of $f_T(v)$ by b, r, b^{-1}, or r^{-1}.

If $r \neq b$ in G_S then the map from tilings to tiling functions is invertible, since we can get the color of any edge in T by comparing f_T at its ends.

It is easy to prove by induction on the number of cells that $f_T(v)$ is well-defined; it is singlevalued since going around any single tile gives a contour word which is equivalent to the identity of G_S. The same observation gives

Proposition 1 (Conway's criterion) *If a polygon P with a colored boundary admits a tiling with a set of tiles S, then its contour word is equivalent to the identity in G_S.*

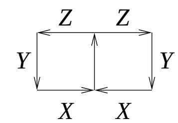
The converse of this proposition is obviously false for some sets of tiles, even for regions consisting of a single cell! This leads us to the notion of an unambiguous set of tiles. A set S is *unambiguous* if the converse of Conway's criterion is true for single cells, i.e. a tile belongs to S if and only if its contour word is $e \in G_S$.

For instance, if S is unambiguous then it cannot contain two tiles which differ only in the color of a single edge unless $S = W$, since dividing the contour word of one of these tiles by the other gives $r = b$.

These and similar considerations easily show that the unambiguous subsets of W are the singletons, W_{12}, W_{13}, W_{23}, W_{16}, W_{25}, W_{34}, W_{56}, W_{123}, W_{124}, W_{1234}, and W. These are the sets of tiles we will focus on next.

Proposition 2 *Conway's criterion is sufficient for a set of tiles S if and only if* **(i)** *S is unambiguous and* **(ii)** *For any three colors x, y, z, there is at least one color w such that the tile with edges x, y, z, w (going, say, counterclockwise) is in S.*

Proof. To prove the first direction, making Conway's criterion sufficient for single squares is the definition of unambiguity. Since the contour word of the domino shown in the figure below is $xx^{-1}y^{-1}z^{-1}zy = e$, sufficiency implies that it must be tileable, meaning that there must exist a color w for the interior edge. To prove the converse, note that if the second condition holds, we can take any polygon, start at the boundary, and repeatedly choose cells which can be removed while keeping the region simply-connected as in Muchnik and Pak [19]. Since each such cell has at most three of its edges set by the boundary conditions, condition (ii) allows us to place a tile there and remove it from the region, until only one cell remains. This cell is tileable if and only if it is in S, which if S is unambiguous means if and only if Conway's criterion holds. □



It is easy to see that this rules out all sets except W_{56}, W_{1234}, and W, for which the reader can easily check both conditions. We discuss these further below.

3 Unambiguous two-color tiles

3.1 Trivial cases

For W_1, W_2, W_3, W_4, W_{12}, W_{13}, W_{23} and W_{123}, a polygon has at most one tiling, and if it exists we can find it in time proportional to the area. This is because for all these sets the tile is determined by the colors of two adjacent edges, so we can start at a corner of P and scan, say, left to right and top to bottom.

Trivially any polygon can be tiled if $S = W$, and Conway's criterion is trivially sufficient with $G_W = \mathbb{Z}_4$. We note as well that the number of such tilings is 2^m where m is the number of edges in the interior of P. If we define a local flip as changing the color of an edge (and the two tiles on either side of it), then the set of tilings has the structure of an m-dimensional hypercube. We will see below that similar structures can be found for other unambiguous sets of tiles.

3.2 Finite tiling groups: W_{56} and W_{1234}

Proposition 2 shows that Conway's criterion is sufficient as well as necessary for W_{56} and W_{1234}. This gives us a linear-time algorithm for tileability: simply calculate P's contour word and compare it to the identity.

The tiling group of W_{56} is $\langle b, r \mid b^3 r, r^3 b \rangle$. Since $b^3 r = e$, we have $r = b^{-3}$, and since $b = r^{-3} = b^9$, the group is isomorphic to $\mathbb{Z}_8$ with (say) $b = 1$ and $r = -3$. For W_{1234}, on the other hand, since $brbr = bbrr = e$ we have $br = rb$, and since $b^4 = r^4 = b^2r^2 = e$, the tiling group is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ with (say) $r = (1, 0)$ and $b = (1, 1)$.

Although their tiling groups are not isomorphic, there is a simple bijection between tilings with the two sets. By flipping the colors of the horizontal edges on the even-numbered rows (say), we change one edge of each tile, transforming tiles in W_{56} to tiles in W_{1234} and vice versa. This may also change some of the colors on the boundary, so this is actually a bijection between pairs (P, T) where P is a region with colored boundary and T is a tiling of it.

There is also a simple bijection between these and assignments of two colors, say yellow and green, to the vertices of P. If we color an edge red if its two vertices are the same color and blue if they are different, we obtain tilings with the set W_{1234}. This also shows that, once the colors on P's boundary are chosen, the number of tilings is 2^k where k is the number of vertices in the interior of P. The local flip changes the color of a vertex and thus the four edges and the four tiles around it, and the graph of tilings forms a k-dimensional hypercube. This gives a trivial algorithm for sampling random tilings, by flipping k independent coins.

3.3 Infinite tiling groups and height functions

The use of height functions for tilings was introduced by Thurston [26], and since then have been applied to several sets of polyominoes and polygons. They had been found independently in statistical physics, and have been applied to ice models, antiferromagnets, and Potts models (see e.g. [3, 6]). We will see that they can be applied to some sets of Wang tiles as well.

The idea is to transform the tiling function f_T to an integer height at each vertex, by composing it with an appropriate function $z : G_S \to \mathbb{Z}$ and writing $h_T = z \circ f_T$. Then we can define a partial order on the set of tilings of a particular polygon with colored boundary,

$$T \preceq T' \iff \forall v \in P : h_T(v) \leq h_{T'}(v)$$

The height function typically possesses the following properties, which will help us solve tiling problems:

- Given the boundary conditions, there is a one-to-one relation between tilings and height functions.
- Local flips can be applied at local extrema of h_T in the interior of P, and can connect any tiling to any other with the same boundary conditions.
- With respect to $\preceq$, the the set of tilings is a distributive lattice. In particular, there are minimal and maximal tilings $\bot$ and $\top$.
- $\bot$ is convex, i.e. $h_\bot$ has no local maxima in the interior of P.

The distributive lattice structure will help us in several ways. Since there exists a tiling iff there exists a minimal tiling and the heights of the vertices of the boundary are given by the boundary conditions, this will give us a straightforward algorithm for tileability. We will also have an algorithm to compute the shortest way to pass from a tiling to another one by flips. Finally, the technique of coupling from the past will allow us to sample random tilings in polynomial time [16, 22].

3.3.1 W_5 and dominoes

It is easy to see that if S is the singleton W_5 we have domino tilings, where blue edges are the boundaries of dominoes and red edges cross their interiors. The tiling group $\langle b, r \mid b^3 r \rangle$ is isomorphic to $\mathbb{Z}$ with $b = 1$ and $r = -3$. Thus we can take the height function $h_T = f_T$ where z is simply the identity. If the perimeter of P is blue, Conway's criterion simply checks that there are an equal number of black and white squares in P. We already know from Proposition 2 that this criterion is not sufficient. To discuss the lattice structure of the set of tilings we will follow [24] and omit the proofs.

A local flip can be applied at a vertex v when its two incoming edges have the same color, and its two outgoing edges have the same color. Equivalently, a flip consists of exchanging two horizontal dominoes for two vertical ones or vice versa. It is easy to see that a flip is possible at v if and only if v is a local extremum of the height function. Since this flip changes the color of all four edges around a vertex, it can only be applied at a vertex in the interior of P. We call a flip *upwards* if it transforms a local minimum to a local maximum and *downwards* if it does the reverse. The reader can check that these flips increase or decrease $h_T(v)$ by 4.

Recall that a *lattice* [5, 9] is a set equipped with a partial order, where any two elements a and b have a unique infimum $a \wedge b$ and a unique supremum $a \vee b$. A lattice is *distributive* if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Viewing the set as a directed acyclic graph, it follows that any pair of directed paths between two points (which, if a path exists, are comparable) have the same length.

Standard arguments then allow us to prove the following properties of the partial order $\preceq$ defined above:

Proposition 3 (Flips and order) *For any pair of tilings T and T' of the same polygon P with the same colored boundary, $T \preceq T'$ if and only if T' can be obtained from T by a sequence of upwards flips.*

Proposition 4 (Lattice structure) *If a polygon P is tileable then, with respect to the order $\preceq$, the graph of tilings is a distributive lattice. In particular, there is a unique minimal tiling $\perp$.*

Proposition 5 (Convexity) *Let P be a tileable polygon and let $\perp$ be its minimal tiling. Then $h_\perp$ has no local maximam in the interior of P.*

Proposition 6 (Flip formula) *For any pair of tilings T and T' with the same boundary conditions, the minimal number of flips to pass from T to T' is $(1/4)\sum_v |h_{T'}(v) - h_T(v)|$.*

Combining Propositions 4 and 5 gives the following algorithm, which either constructs the minimal tiling or confirms that the region is not colorable:

- Calculate the heights of vertices on the boundary. If Conway's criterion is not satisfied then the region is not tileable. Otherwise repeat the following steps until the region is completely tiled.

- Whenever all four corners of a cell have an assigned height, place the appropriate tile there and remove that cell from the region. If no tile is consistent with these heights, halt and conclude that the region is not tileable.

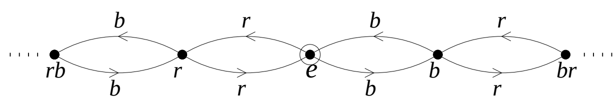
- Find a vertex v with maximum height $h_{\max}$ on the current boundary; it has a neighbor w whose height has not yet been assigned. Since $h_\perp$ may not have local maxima in the interior, set $h_\perp(w)$ smaller than $h_{\max}$, either to $h_{\max} - 1$ or $h_{\max} - 3$ depending on the orientation of the edge from v to w.

If we remove cells at P's boundary whenever one of their edges is red, tilings with W_5 correspond exactly to domino tilings of the remaining region. We can then use the results of Propp and Wilson [22] and Luby, Randall and Sinclair [16] to sample perfectly random tilings in time polynomial in the area of P.

3.3.2 W_{34}, W_{124}, and Eulerian orientations

The tiling group of W_{34} is $\langle b, r \mid brbr, b^2r^2 \rangle$, which is isomorphic to $\mathbb{Z}$ with $b = +1$ and $r = -1$. Once again we can take $h_T = f_T$ as our height function. We have all the same tools as in the previous example, except that now for any pair of neighbors u, v we have $|h_T(u) - h_T(v)| = 1$. This is recognizable as the height function for Eulerian orientations of the dual lattice, called the *six-vertex ice model* by physicists [3]. This is also equivalent to the height function for three-colorings of the square lattice, and to alternating-sign matrices [20]. The algorithm for tileability is completely analogous to that for the domino tiling W_5, except that we always set the height of the neighboring vertex to $h_{\max} - 1$. The progress of the algorithm is shown in Figure 1; it either constructs the minimum tiling, or arrives at a contradiction where two neighboring vertices have heights differing by more than 1, violating the definition of the height function and proving that the region is not tileable.

For W_{124}, the tiling group $\langle b, r \mid b^4, r^4, b^2 r^2 \rangle$ is more complex. However, by imposing the additional relations $b^2 = r^2 = e$ we can obtain a simple quotient for it, the free group on two generators of order 2, which has the following Cayley graph:



While this is not isomorphic to $\mathbb{Z}$ it clearly has the same "shape" as $\mathbb{Z}$. We can define a height function $h_T = z \circ f_T$ with the following z, taking advantage of the fact that if $r^2 = b^2 = e$ every element can be written as a word w of alternating r's and b's:

$$
\begin{aligned}
z(w) &= |w| \text{ if } w \text{ begins with } b \\
z(w) &= -|w| \text{ if } w \text{ begins with } r
\end{aligned}
$$

Then the same results follow as for W_{34}.

Just as for W_{56} and W_{1234}, there is a simple bijection between tilings with W_{34} and those with W_{124} even though their tiling groups are not isomorphic. If we flip the colors of all the horizontal edges (say), each tile in W_{34} becomes one in W_{124} and vice versa. Composing this with the bijections shown above gives a simple bijection between W_{124} tilings and Eulerian orientations.

As in the previous case, the techniques of [16, 22] can be used to sample random tilings in polynomial time.

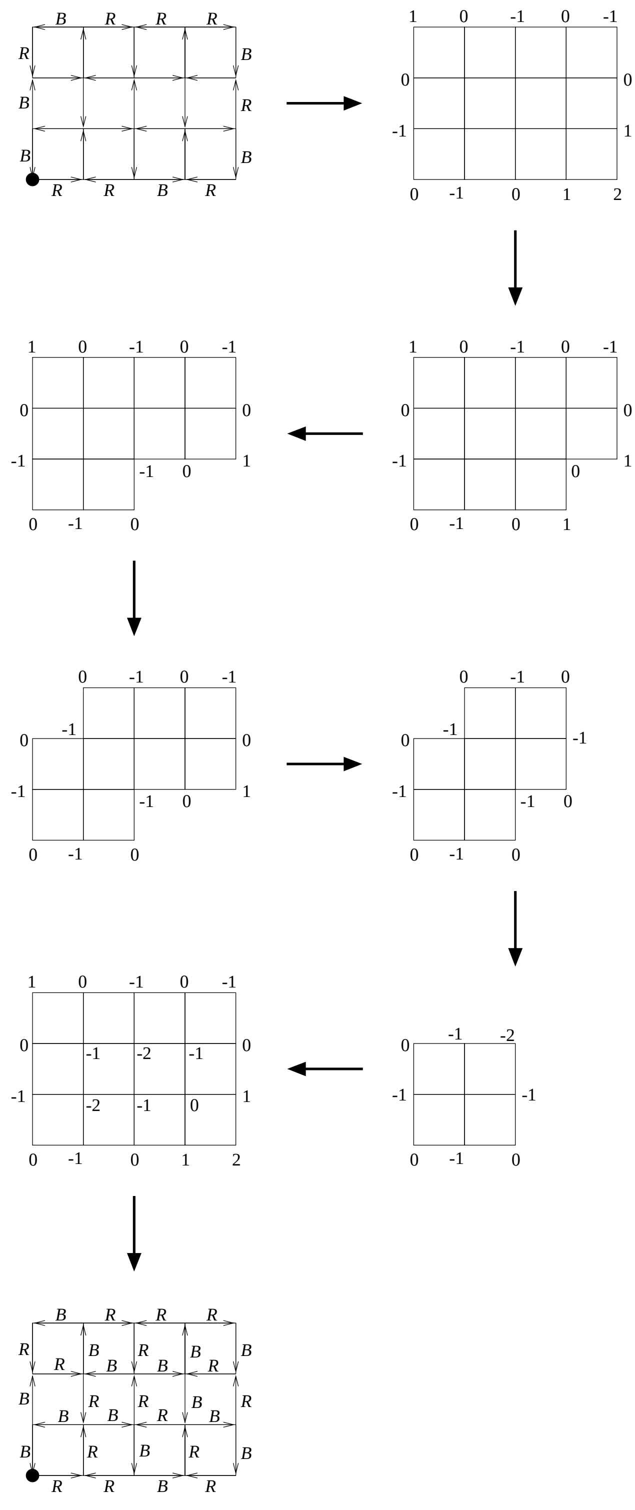


Figure 1: The progress of the tiling algorithm for W_{34}, or equivalently Eulerian orientations of the grid, which constructs the minimal tiling or shows that none exists.

3.4 The curious case of W_{16}

The set W_{16} (and its symmetry partner W_{25}) is the only unambiguous two-color set which remains unsolved. Its tiling group $\langle b, r \mid b^4, r^3 b \rangle$ is isomorphic to $\mathbb{Z}_{12}$ with $b = 3$ and $r = -1$. Thus its tiling group is finite; however, Conway's criterion is not sufficient.

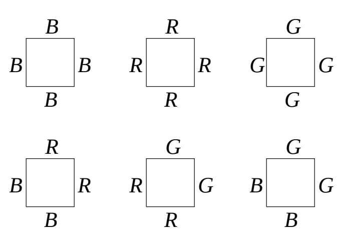
By lifting from $\mathbb{Z}_{12}$ to $\mathbb{Z}$ we see that the number of w_1 tiles on white squares minus the number of w_1 tiles on black squares is an invariant, since for a polygon P this is $n/12$ where n is the integer corresponding to P's contour word. Notice that, for each vertex, the tiling function has three possible values, since $f_T(v)$ is equivalent mod 4 to the length of a path from the origin vertex to v.

We leave as an open problem whether there is a polynomial-time algorithm to tell whether a given polygon can be tiled with W_{16}. The reader may enjoy showing that a region with red boundary is tileable if and only if it can be tiled by dominoes and X-pentominoes. It seems likely that such tilings are NP-complete for non-simply-connected regions using constructions similar to Moore and Robson [18].

4 Examples with more colors

4.1 Height functions on Cayley trees

In this section we show that the notion of height function can also be used for some Wang tiles with three (or more) colors. While the height function is more complex, it still gives us an efficient algorithm to determine whether a region is tileable. Our example is a generalization of W_{124}, where each tile has at most two colors, and where every colored edge shares a vertex with another edge with the same color. The set V is the following:



Taking our colors to be Red, Green and Blue, with the associated generators r, g and b, then the tiling group is $\langle r, g, b \,|\, r^2g^2, g^2b^2, b^2r^2 \rangle$ (note that these relations imply $r^4 = g^4 = b^4 = e$) and this appears to be quite complex. Luckily, there is a simple quotient which does not create any ambiguity, namely $G = \langle r, g, b \,|\, r^2, g^2, b^2 \rangle$. Its Cayley graph $\Gamma(G)$ has a tree structure (if opposite arrows are identified) as shown in Figure 2, and a height function can be constructed from the following axioms:

- For every $n \geq 0$, $z((br)^n) = -2n$ and $z((br)^n b) = -2n - 1$.
- For each element x of G, there exists a unique neighbor $p_G(x)$ of x, called the G-*predecessor* of x, such that $z(p_G(x)) = z(x) - 1$. For the other two neighbors y of x, we have $z(y) = z(x) + 1$.

These imply, for instance, that if w is a word in $\{r, g, b\}$ where no two adjacent letters are the same, then

$$\begin{aligned} z(w) &= |w| \text{ if } w \text{ begins with } r \text{ or } g \\ z(w) &= |w| - 4n \text{ if } w \text{ begins with } (br)^n g \\ z(w) &= |w| - 4n - 2 \text{ if } w \text{ begins with } (br)^n bg \end{aligned}$$

To define a partial order on G, we say that $x \leq_G y$ if there exists a finite sequence of elements of G, starting with x and finishing with y, such that the predecessor in the sequence is the G-predecessor as defined above. If we define the index of any element $x \in G$ as $h(x)$ mod 2, then the partial order $\leq_G$ induces an order $\leq_i$ on each of the two index classes. Each element v has a unique predecessor in its index class, $p_i(x) = p_G(p_G(x))$.

For each pair x, y of elements with the same index i, we define $\inf_i(x, y)$ as the infimum of x and y with respect to $\leq_i$. Notice that $\inf_i(x, y)$ is not always equal to the infimum $\inf_G(x, y)$ with respect to $\leq_G$, since the latter might be in the other index class, in which case $\inf_i(x, y) = p_G(\inf_G(x, y))$. Note also that $h_T(v)$ is equivalent mod 2 to the length of any path from the origin vertex to v, so $f_T(v)$ and $f_{T'}(v)$ are in the same index class for any two tilings T, T' of the same region.

A local flip changes the colors around a fixed interior vertex v. This can only happen if all the edges linking v to its neighbors have the same

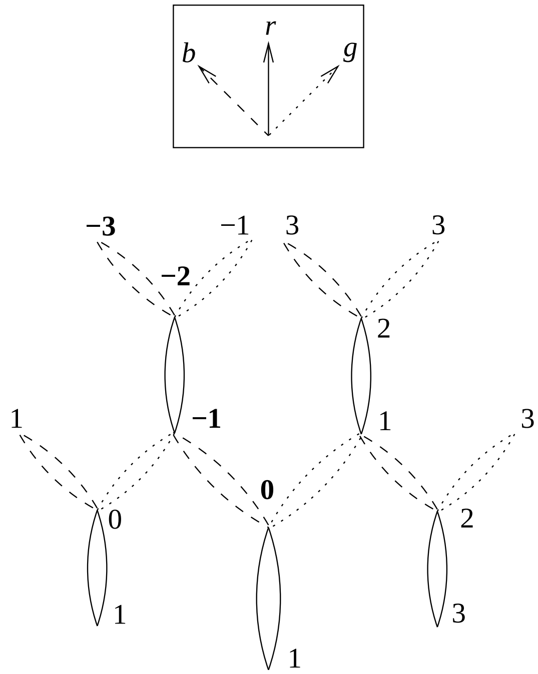


Figure 2: The Cayley tree $\Gamma(G)$ for the three-color tiling, and the height function z.

color, which means that the height function has a local extremum at v.

As before, we define a partial order on tilings by $T \preceq T'$ if $f_T(v) \leq_G f_{T'}(v)$ for all v (which implies $f_T \leq_i f_{T'}$ and $h_T \leq h_{T'}$). We also define a distance between elements of G: for $x, y \in G$, let $d(x, y)$ be the length of the shortest path between them, using the generators r, g and b. Then define the distance between two tilings as $d(T, T') = \sum_v d(f_T(v), f_{T'}(v))$. Note that if T and T' are comparable, we have $d(T, T') = \sum_v |h_T(v) - h_{T'}|$.

Proposition 7 (Flips and order) *For any pair of tilings T and T' with the same boundary conditions, $T \preceq T'$ if and only if T can be obtained from T' by a sequence of downwards flips.*

Proof. By induction on $d(T, T')$. Take a vertex v such that $h_{T'}(v) > h_T(v)$, where v is a local maximum of $h_{T'}$ (note v is an interior vertex). Then $h_{T'}(u) = h_{T'}(v) - 1$ and $f_{T'}(u) = p_G(f_{T'}(v))$ for all neighbors u of v. Thus T' can be flipped downwards at v, inducing a tiling T'' such that $f_{T''}(v) = p_i(f_T(v))$ where i is the index of $f_T(v)$. We have $T \preceq T'' \preceq T'$, and $d(T, T'') = d(T, T') - 2$. This gives the result by induction, until $d(T, T'') = 0$ and $T = T''$. □

Recall that an *inferior semi-lattice* is similar to a lattice, but with only the infimum of two elements $a \wedge b$ guaranteed to be unique. Then:

Proposition 8 (Inferior semi-lattice structure) *If a polygon P is tileable then the graph of tilings with respect to $\preceq$ is a inferior semi-lattice, where the tiling function of $T'' = T \wedge T'$ is given by $f_{T''} = \inf_i(f_T, f_{T'})$ at each vertex.*

Proof. We have to prove that $f_{T''} = \inf_i(f_T, f_{T'})$ is a valid tiling function. Note that if u, v are neighbors (by which we mean that the edge connecting them is in P) then their values of the tiling function are neighbors in $\Gamma(G)$, i.e. either $f_T(u) = p_G(f_T(v))$ or $f_T(v) = p_G(f_T(u))$, and similarly for T'. Therefore, we need to show that $f_{T''}(u)$ and $f_{T''}(v)$ are neighbors in $\Gamma(G)$.

We have two cases up to symmetry. If $f_T(u) = p_G(f_T(v))$ and $f_{T'}(u) = p_G(f_{T'}(v))$, then if $f_T(v)$ and $f_{T'}(v)$ are comparable, then $f_{T''}(u) = p_G(f_{T''}(v)$. If $f_T(v)$ and $f_{T'}(v)$ are incomparable, then $\inf_G(f_T(u), f_{T'}(u)) = \inf_G(f_T(v), f_{T'}(v))$, in which case either $f_{T''}(u) = p_G(f_{T''}(v))$ or $f_{T''}(v) = p_G(f_{T''}(u))$ using the relationship between $\inf_G$ and $\inf_i$ stated above. The other case, in which $f_T(u) = p_G(f_T(v))$ and $f_{T'}(v) = p_G(f_{T'}(u))$, can be analyzed similarly.

Thus the pointwise infimum of f_T and $f_{T'}$ with respect to $\leq_i$ is a tiling function, and the corresponding tiling T'' is clearly the unique infimum of T and T' with respect to $\preceq$. □

This also implies that there is a unique minimal tiling $\perp$, which has the same properties as in the simpler cases above:

Proposition 9 (Convexity) *Let P be a region tileable with tiles in V and let $\perp$ be its minimal tiling. Then $h_\perp$ has no local maxima in the interior of P.*

Proof. Suppose that for $\perp$ we have an internal vertex v such that $h_\perp(v)$ is a local maximum. Then flipping v downwards would give a new tiling $T' \prec \perp$, a contradiction. □

This gives us an efficient algorithm for constructing the minimal tiling and confirming tileability similar to that of Section 3.3.1, except that we set $f_\perp(w) = p_G(f_{\max})$. We also have:

Proposition 10 (Flip formula) *For any pair of tilings T and T' satisfying the same boundary condition, the minimal number of flips to go from T to T' is $(1/2)d(T,T')$.*

Proof. We use the method of [25]. Clearly, $(1/2)d(T,T')$ is a lower bound for the number of necessary flips since flipping at v changes $d(T,T')$ by zero or ± 2. Now take an interior vertex v such that $f_{T'}(v) \neq f_T(v)$ and $\sup(h_T(v), h_{T'}(v))$ is locally maximal. We assume w.l.o.g. that $\sup(h_T(v), h_{T'}(v)) = h_T(v)$, in which case $f_T(u) = p_G(f_T(v))$ for each neighbor u of v. Then T can be flipped at v, moving $f_T(v)$ towards $f_{T'}(v)$ in $\Gamma(G)$ and giving a tiling T'' such that $d(T'',T') = d(T,T')-2$ (notice that this flip either reduces the height of v or keeps it the same, changing $f_T(v)$ but not $h_T(v)$). This gives the result by induction. □

Note that we have actually defined one height function in an uncountably infinite family of them, where the height decreases along one path (in this case $(br)^*$) and increases along all others. Each of these induces a different partial order, and for each computable one we have an algorithm similar to that above to find the minimal tiling with respect to it.

4.2 Higher-dimensional height functions

As another example, consider the set V of four-color Wang tiles where a tile is in V if and only if each color appears once on its boundary. The tiling group $\mathbb{Z}^4/\{1,1,1,1\}$ is Abelian and infinite, and is isomorphic to the body-centered cubic lattice with the four generators $(+1,+1,+1)$, $(+1,-1,-1)$, $(-1,+1,-1)$, and $(-1,-1,+1)$. This corresponds to a three-dimensional height function. Similarly, three-color triangular tiles have a two-dimensional height function $\mathbb{Z}^3/\{1,1,1\}$ isomorphic to the triangular lattice, and six-color hexagonal tiles have a five-dimensional height function.

All these tilings are equivalent to edge k-colorings of the dual lattice (the square, hexagonal, and triangular lattices respectively) where k is equal to the dual lattice's degree. Edge 3-colorings of the hexagonal lattice are also equivalent to vertex 4-colorings of the triangular lattice, and were studied by Baxter [2], Huse and Rutenberg [12], and Moore and Newman [17]. Edge 4-colorings of the square lattice were studied by Kondev and Henley [14]. None of these tilings are connected under local moves, but they are connected under "loop moves" where we choose two colors, find a loop consisting of edges with those two colors, and switch the colors along the loop. Little is known about the mixing time of the resulting Markov chain; the techniques of [16, 22] appear not to apply, since these non-local moves make it hard to define a monotonic coupling. We suggest this as an area for future research.

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