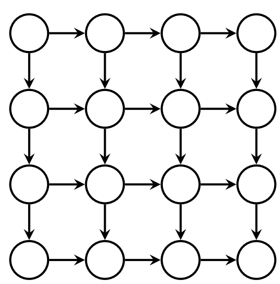
CS 362, HW 7

Prof. Jared Saia, University of New Mexico

1. A cat wants to get from the top left post in a n by n grid to the bottom right post. For $i, j \in [1, n]$, from post (i, j) there is a directed edge to the post $(i + 1, j)$ if $i < n - 1$, and a directed edge to post $(i, j + 1)$, if $j < n$. Each edge has a probability that gives the probability the cat would succeeds in jumping that edge. The probability that the cat succeeds along a path is the product of the edge probabilities on that path. Write a dynamic program that finds the maximum probability of success for any path from the top left post to the bottom right. Explain how you would augment your dynamic program to return one path that achieves this maximum probability.



2. There are n people and n chairs at a dinner party. Each person i has an assigned chair i, for $i \in [1, n]$. Unfortunately, person 1 is drunk. They sit at a chair chosen uniformly at random. Then for $i \in [2, n]$, person i first checks chair i. If it's empty, they sit there. If it's full they select a chair chosen uniformly at random from all empty chairs.

   Let $f(n)$ be the probability that when there are n people, person n sits in their assigned chair.

   (a) Note that $f(1) = 1$. What is $f(2)$?

   (b) Write a recurrence relation for $f(n)$. Hint: Person 1 sits at chair 1 with probability $1/n$. What is $f(n)$ in this case? What if person

1 sits at chair j? What will people 2 through $j-1$ do? Can you now say the probability will be $f(x)$ for some $x < n$. The right hand side of your recurrence should contain many $f(x)$ values, one for each place the first person may sit.

(c) Use guess and check and proof by induction to solve your recurrence for the exact value of $f(n)$ for $n \geq 2$. Don't forget to include the base case, IH, and IS.

3. There is a grid of n nodes, where n is a perfect square, with $\sqrt{n}$ rows and $\sqrt{n}$ columns.

   (a) Assume each node fails independently with probability p. What is the expected number of nodes that fail?

   (b) Use Markov's inequality to bound the probability that at least 1 node fails. Hint: Let X be the number of failed nodes, use $E(X)$ from above.

   (c) Use a union bound to bound the probability that no node fails. Hint: Let E_i be the event that node i fails. Bound $Pr(\cup_{i=1}^{n} E_i)$.

4. In the grid from the previous problem, a row is said to be *bad* if any of the nodes in the row fail.

   (a) What is the expected number of bad rows?

   (b) Now assume that there are exactly $n/2$ failed nodes, uniformly distributed among all nodes. What is the expected number of bad rows?