CS 561, Lecture 4

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Outline

- Loop Invariants
- Heaps

Correctness of Algorithms

- The most important aspect of algorithms is their correctness
- An algorithm by definition *always* gives the right answer to the problem
- A procedure which doesn't always give the right answer is a *heuristic*
- All things being equal, we prefer an algorithm to a heuristic
- How do we prove an algorithm is really correct?

Loop Invariants

- A useful tool for proving correctness is loop invariants.
- Loop Invariants are essentially proof by induction

Loop Invariants

Three things must be shown about a loop invariant

- **Initialization:** Invariant is true before first iteration of loop (Base Case)
- **Maintenance:** If invariant is true before iteration i, it is also true before iteration $i+1$ (Inductive Step)
- **Termination:** When the loop terminates, the invariant gives a property which can be used to show the algorithm is correct (Wrapup)

Example Loop Invariant

- We'll prove the correctness of a simple algorithm which solves the following interview question:
- *Find the middle of a linked list, while only going through the list once*
- The basic idea is to keep two pointers into the list, one of the pointers moves twice as fast as the other
- (Call the head of the list the 0-th elem, and the tail of the list the $(n-1)$-st element, assume that $n-1$ is an even number)

Example Algorithm

```
GetMiddle (List l){
  pSlow = pFast = l;
  while ((pFast->next)&&(pFast->next->next)){
    pFast = pFast->next->next
    pSlow = pSlow->next
  }
  return pSlow
}
```

Example Loop Invariant

- *Invariant: At the start of the i-th iteration of the while loop, pSlow points to the i-th element in the list and pFast points to the $2i$-th element*
- **Initialization:** True when $i = 0$ since both pointers are at the head
- **Maintenance:** if pSlow, pFast are at positions i and $2i$ respectively before i-th iteration, they will be at positions $i+1$, $2(i+1)$ respectively before the $i+1$-st iteration
- **Termination:** When the loop terminates, pFast is at element $n-1$. Then by the loop invariant, pSlow is at element $(n-1)/2$. Thus pSlow points to the middle of the list

Challenge

- Figure out how to use a similar idea to determine if there is a loop in a linked list *without marking nodes!*

What is a Heap

- "A heap data structure is an array that can be viewed as a nearly complete binary tree"
- Each element of the array corresponds to a value stored at some node of the tree
- The tree is completely filled at all levels except for possibly the last which is filled from left to right

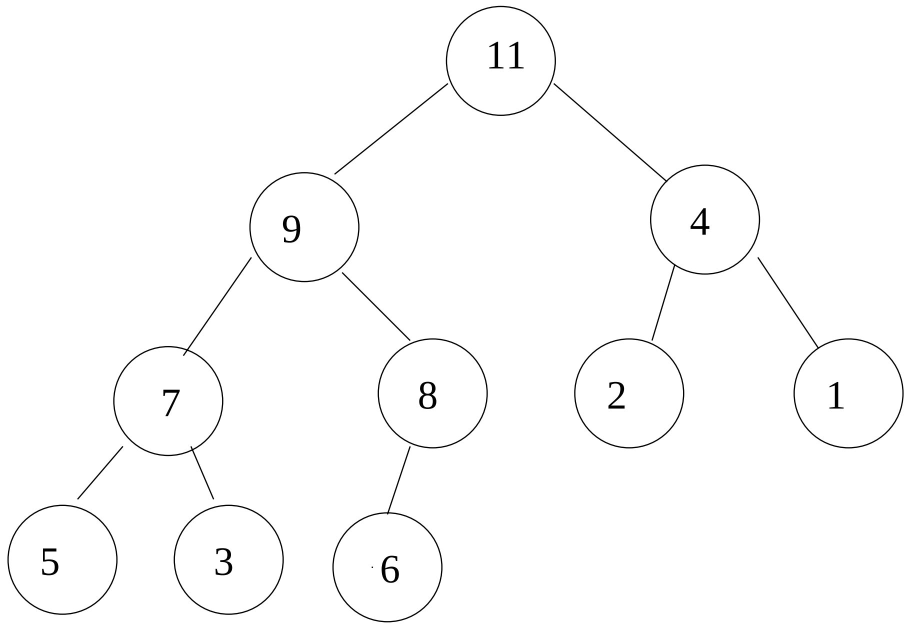
heap-size (A)

- An array A that represents a heap has two attributes
  - length (A) which is the number of elements in the array
  - heap-size (A) which is the number of elems in the heap stored within the array
- I.e. only the elements in A[1..heap-size (A)] are elements of the heap

Tree Structure

- A[1] is the root of the tree
- For all i, $1 < i <$ heap-size (A)
  - Parent (i) $= \lfloor i/2 \rfloor$
  - Left (i) $= 2i$
  - Right (i) $= 2i + 1$
- If Left (i) $>$ heap-size (A), there is no left child of i
- If Right (i) $>$ heap-size (A), there is no right child of i
- If Parent (i) < 0, there is no parent of i

Example



A:

1	2	3	4	5	6	7	8	9	10
11	9	4	7	8	2	1	5	3	6

Max-Heap Property

- For every node i other than the root, A[Parent (i)] $\geq$ A[i]

Max-Heap Property

- For every node i other than the root, A[Parent (i)] $\geq$ A[i]
- Parent is always at least as large as its children
- Largest element is at the root

(A Min-heap is organized the opposite way)

Height of Heap

- Height of a node in a heap is the number of edges in the longest simple downward path from the node to a leaf
- Height of a heap of n elements is $\Theta(\log n)$. Why?

Maintaining Heaps

- Q: How to maintain the heap property?
- A: *Max-Heapify* is given an array and an index i. Assumes that the binary trees rooted at $Left(i)$ and $Right(i)$ are max-heaps, but $A[i]$ may be smaller than its children.
- *Max-Heapify* ensures that after its call, the subtree rooted at i is a Max-Heap

Max-Heapify

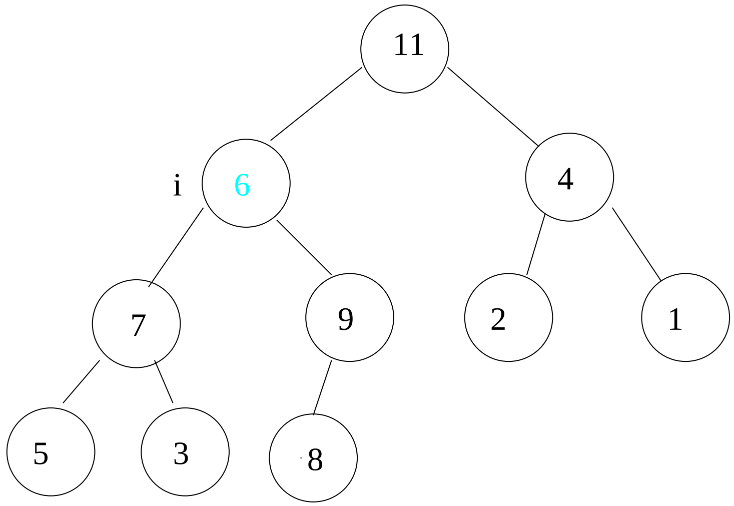
- Main idea of the Max-Heapify algorithm is that it percolates down the element that start at $A[i]$ to the point where the subtree rooted at i is a max-heap
- To do this, it repeatedly swaps $A[i]$ with its largest child until $A[i]$ is bigger than both its children
- For simplicity, the algorithm is described recursively.

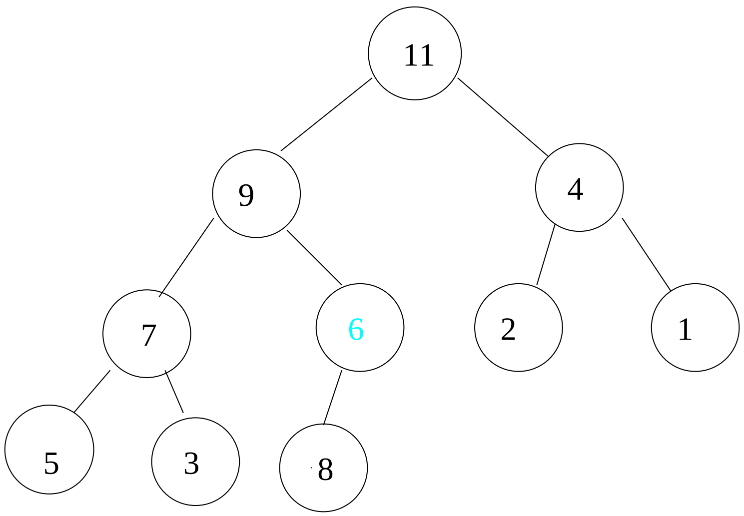
Max-Heapify

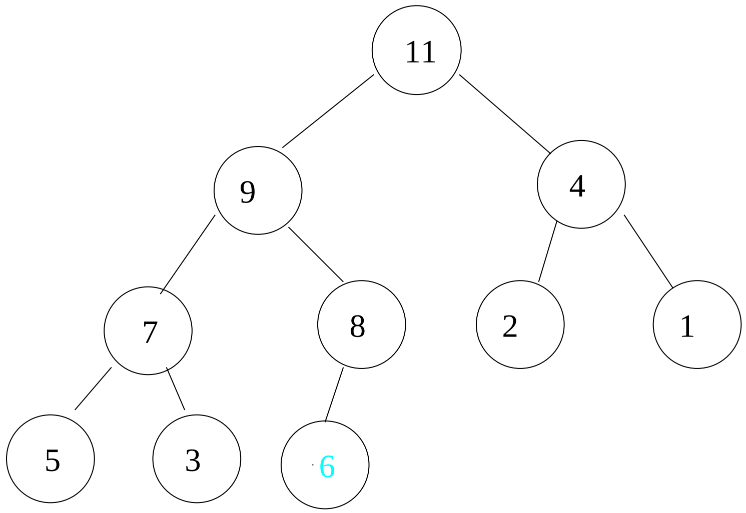
Max-Heapify (A,i)

1. $l = Left(i)$
2. $r = Right(i)$
3. $largest = i$
4. if ($l \leq$ heap-size(A) and $A[l] > A[i]$) then $largest = l$
5. if ($r \leq$ heap-size(A) and $A[r] > A[largest]$) then $largest = r$
6. if $largest \neq i$ then
   (a) exchange A[i] and A[largest]
   (b) Max-Heapify (A,largest)

Example







Analysis

- Let $T(h)$ be the runtime of max-heapify on a subtree of height h
- Then $T(1) = \Theta(1)$, $T(h) = T(h - 1) + 1$
- Solution to this recurrence is $T(h) = \Theta(h)$
- Thus if we let $T(n)$ be the runtime of max-heapify on a subtree of *size* n, $T(n) = O(\log n)$, since $\log n$ is the maximum height of heap of size n

Build-Max-Heap

- Q: How can we convert an arbitrary array into a max-heap?
- A: Use Max-Heapify in a bottom-up manner
- Note: The elements A[$\lfloor n/2 \rfloor + 1$],..,A[n] are all leaf nodes of the tree, so each is a 1 element heap to begin with

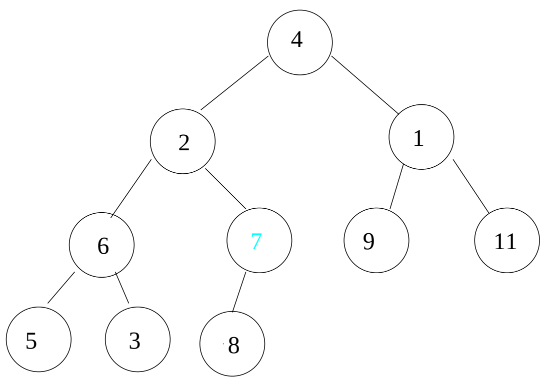
Build-Max-Heap

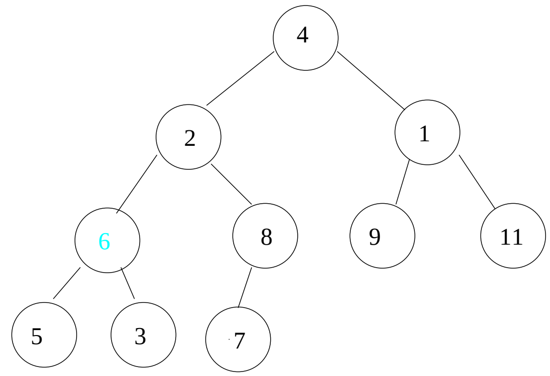
Build-Max-Heap (A)

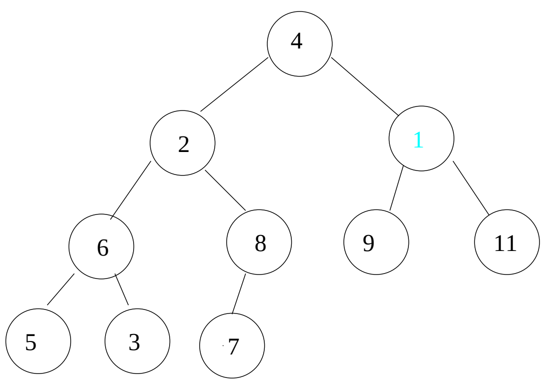
1. heap-size (A) = length (A)
2. for ($i = \lfloor length(A)/2 \rfloor$;$i >$ 0;$i --$)
   (a) do Max-Heapify (A,i)

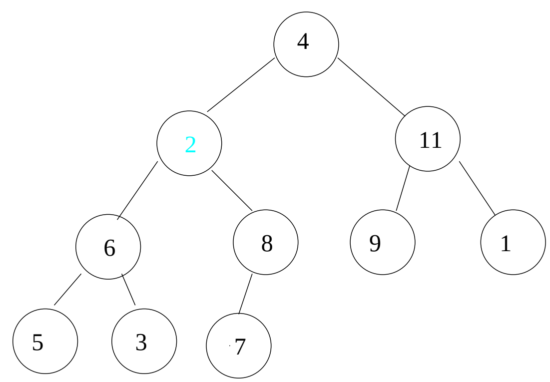
Example

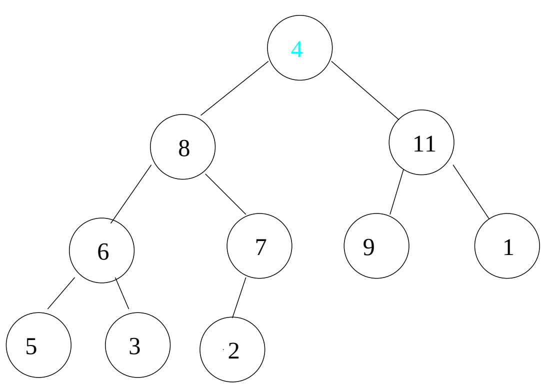
A = 4 2 1 6 7 9 11 5 3 8

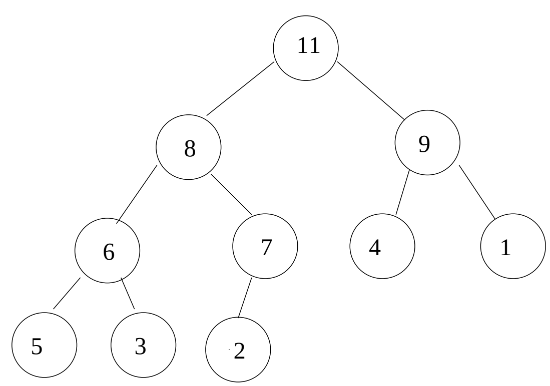












Loop Invariant

- Loop Invariant: "At the start of each iteration of the for loop, each node $i + 1, i + 2, \ldots n$ is the root of a max-heap"

Correctness

- **Initialization:** $i = \lfloor n/2 \rfloor$ prior to first iteration. But each node $\lfloor n/2 \rfloor + 1$, $\lfloor n/2 \rfloor + 2, \ldots, n$ is a leaf so is the root of a trivial max-heap
- **Termination:** At termination, $i = 0$, so each node $1, \ldots, n$ is the root of a max-heap. In particular, node 1 is the root of a max heap.

Maintenance

- **Maintenance:** First note that if the nodes $i+1, \ldots n$ are the roots of max-heaps before the call to Max-Heapify (A,i), then they will be the roots of max-heaps after the call. Further note that the children of node i are numbered higher than i and thus by the loop invariant are both roots of max heaps. Thus after the call to Max-Heapify (A,i), the node i is the root of a max-heap. Hence, when we decrement i in the for loop, the loop invariant is established.

Time Analysis

(Naive) Analysis:

- Max-Heapify takes $O(\log n)$ time per call
- There are $O(n)$ calls to Max-Heapify
- Thus, the running time is $O(n \log n)$

Time Analysis

Better Analysis. Note that:

- An n element heap has height no more than $\log n$
- There are at most $n/2^h$ nodes of any height h (to see this, consider the min number of nodes in a heap of height h)
- Time required by Max-Heapify when called on a node of height h is $O(h)$.
- Thus total time is: $\sum_{h=0}^{\log n} \frac{n}{2^h} O(h)$

Analysis

$$\sum_{h=0}^{\log n} \frac{n}{2^h} O(h) = O\left(n \sum_{h=0}^{\log n} \frac{h}{2^h}\right) \quad (1)$$

$$= O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h}\right) \quad (2)$$

$$= O(n) \quad (3)$$

The last step follows since for all $|x| < 1$,

$$\sum_{i=0}^{\infty} i x^i = \frac{x}{(1-x)^2} \qquad (4)$$

Can get this equality by recalling that for all $|x| < 1$,

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x},$$

and taking the derivative of both sides!

Heap-Sort

Heap-Sort (A)

1. Build-Max-Heap (A)
2. for (i=length (A);$i > 1$;$i - -$)
   (a) do exchange A[1] and A[i]
   (b) heap-size (A) = heap-size (A) - 1
   (c) Max-Heapify (A,1)

Analysis

- Build-Max-Heap takes $O(n)$, and each of the $O(n)$ calls to Max-Heapify take $O(\log n)$, so Heap-Sort takes $O(n \log n)$
- Correctness???