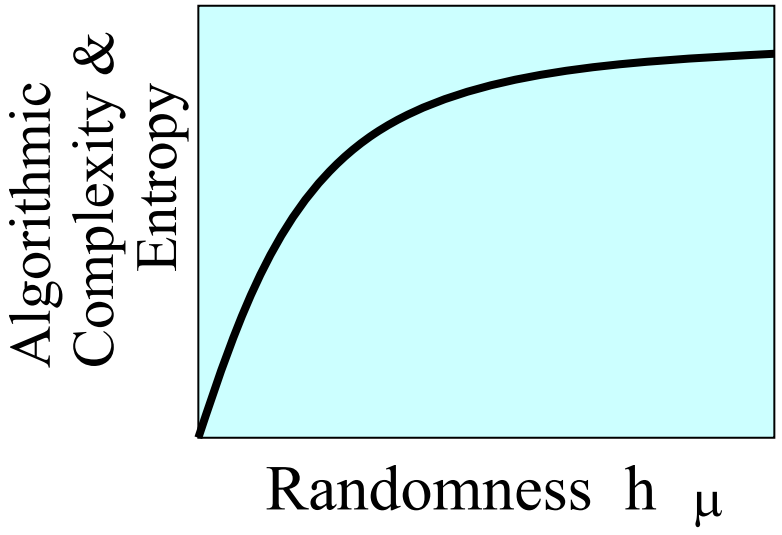
Measures of Complexity

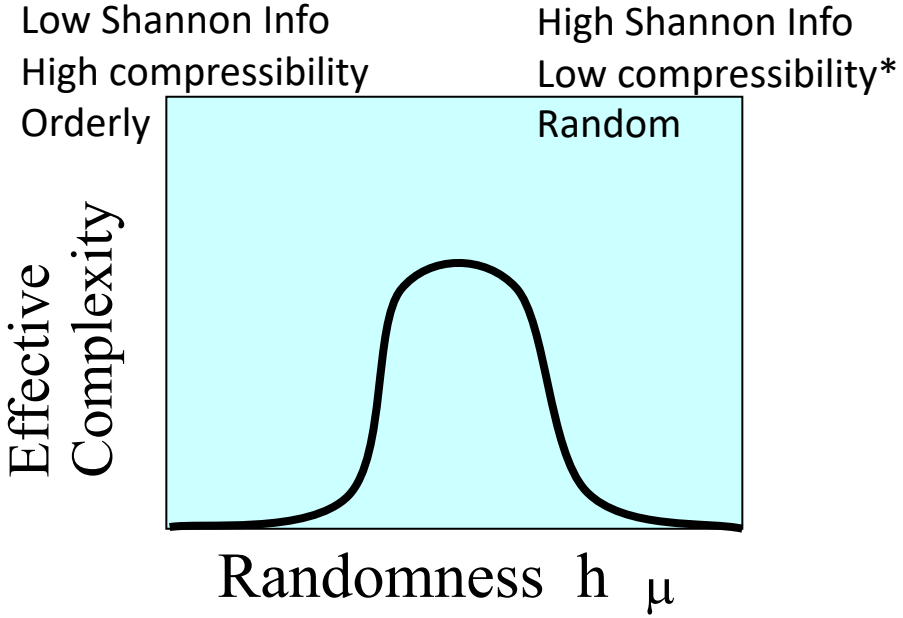
- Entropy
- AIC (Kolmogorov Complexity)
- EFFECTIVE COMPLEXITY
- Total Information
- Logical Depth
- Statistical Complexity
- Size
- Fractal Dimension

Gell-Mann's Effective Complexity

- The length of the shortest description of a set's regularities
- EC(x) = K(r) where r is the set of regularities in x

  and Kolmogorov Complexity (or AIC), K(r), is the length of a concise description of a set
- Highest for entities that are not strictly regular or random





Algorithmic Complexity (AIC)

(also known as Kolmogorov-Chaitin complexity)

- *Kolomogorov* complexity or Algorithmic Information Content (AIC), written *K(x),* is the length, in bits, of the smallest program that when run on a Universal Turing Machine outputs (prints) *x* and then halts.
- Example: What is *K(x)*
  - where *x* is the first 10 even natural numbers?
  - where *x* is the first 5 million even natural numbers?

- Possible representations where *n* is the length of *x*
  - *0, 2, 4, 6, 8, 10, 12, 14, 16, 18, … (2n - 2)*
  - for (j = 0; j < n: j++) printf("%d\n", j * 2)

Algorithmic Complexity (AIC)

(also known as Kolmogorov-Chaitin complexity)

- *Kolomogorov* complexity or Algorithmic Information Content (AIC), written *K(x),* is the length, in bits, of the smallest program that when run on a Universal Turing Machine outputs (prints) *x* and then halts.
- Example: What is *K(x)*
  - where *x* is the first 10 even natural numbers?
  - where *x* is the first 5 million even natural numbers?

- Possible representations where *n* is the length of *x*
  - *0, 2, 4, 6, 8, 10, 12, 14, 16, 18, … (2n - 2)* *K(x) = O(n log n) bits*
  - for (j = 0; j < n: j++) printf("%d\n", j * 2) *K(x) = O(log n) bits **

**Complexity of program size (length), not program running time*

Algorithmic Complexity (AIC)

(also known as Kolmogorov-Chaitin complexity)

- AIC formalizes what it means for a set of numbers to be compressible
  - Data that are redundant can be compressed and have lower AIC.
  - Random strings are incompressible, therefore contain no regularities to compress
    - *K(x)* = | Print(x) |
- Implication: The more random a system, the greater its AIC (and greater entropy)
- Contrast with Statistical simplicity

  Random strings are simple because you can approximate them statistically
  - Coin toss, random walks, Gaussian (normal distributions)
  - You can compress random numbers with statistical descriptions and only a few parameters

- s1 = 1111111111111111
  for i:=1:16
  print '1'

  K(s) is O(log n) where n = length(s1)
  The dominant term is the number "16" whose representation in bits will require $\log_2 16$ = 4 bits

- S2 is a string of 1's, length(s2) = 1 billion
  for i:=1:1,000,000,000
  print '1'

K(s1) is O(log n) where n = length(s2)
K(s2) is O($\log_2 10^9$) (approximately 30 bits)

Constant terms such as the representation of the print statement are ignored

The minimum K(x), given length(x) = n, is log(n).

s3 = 0101010101010101
s4 = 011011011011011
s5 = 0001000010000100001
s6 = 011101110110001101101101

s3 = 0101010101010101
for i:=1:8
print ‘01’
2 bits are needed to represent the pattern, 3 bits to represent the 8 repetitions
K(s3) = O(log(n))

s4 = 011011011011011
for i:=1:5
print ‘011’
3 bits to represent the repeating pattern, log(5) bits to represent the number of repetitions
K(s4) = O(log(n))

s5 = 000100001000100001
for c = 1:2
print ‘000100001’

8 bits to represent the repeating pattern, 1 bit to represent the repetitions. Here the length of the pattern is ½ n, so
K(s5) = O(n)

s6 = 011101110110001101101101

There are two problems with AIC

- Calculation of K(x) depends on the machine we have available (e.g., what if we have a machine with an instruction “print the first 10 even natural numbers”?)
  - COMPLEXITY DEPENDS ON CONTEXT
- Determining K(*x*) for arbitrary *x* is uncomputable

Problem 1: K(x) depends on the programming language

Resolution: optimal specification functions can be defined so that

"The complexity of an object *x* is invariant (up to an additive constant independent of *x*) under transition from one optimal specification function to another."

(Li & Vitanyi "An Introduction Kolmogorov Complexity and it's Applications" 2008)

http://jeremykun.com/2012/04/21/kolmogorov-complexity-a-primer/

Lemma: For any strings w, x and any language L, we have $K_L(w|x) \leq |w| + c$ for some constant c independent of w, x, and $K_L(w) \leq |w| + c'$ for some constant c' independent of w.

Proof. The program which trivially outputs the desired string has length $|w| + c$, for whatever constant number of letters c is required to dictate that a string be given as output. This is clearly independent of the string and any input. □

It is not hard to see that this definition is invariant under a choice of language L up to a constant factor. In particular, let w be a string and fix two languages L, L'. As long as both languages are *universal*, in the sense that they can simulate a universal Turing machine, we can relate the Kolmogorov complexity of w with respect to both languages. Specifically, one can write an interpreter for L in the language of L' and vice versa. We saw a

Problem 2: Identifying the shortest program to print the string is uncomputable

Resolution: None

It is not possible to determine the amount of randomness in any arbitrary string

Regularities can be difficult to identify

- s6 = 011101110110001101101 01
- It looks like a random string, so K(s6) = length(s6)

But its not !

- What is the shortest program that would produce this string?

Fibonacci Series

0 1 1 10 11 101 1000 1101 10101

0 1 1 2 3 5 8 13 21

A short computer program (of length (L_{fb}) that produces the Fibonacci series can generate s6, so K(s6) can be reduced to max (L_{fb}, log(n) where n specifies the length of the Fibonacci series to be printed.

What About

s7=00100100001111110110101010001000100001011010001
100001000110100...?

This is a statistically random string.

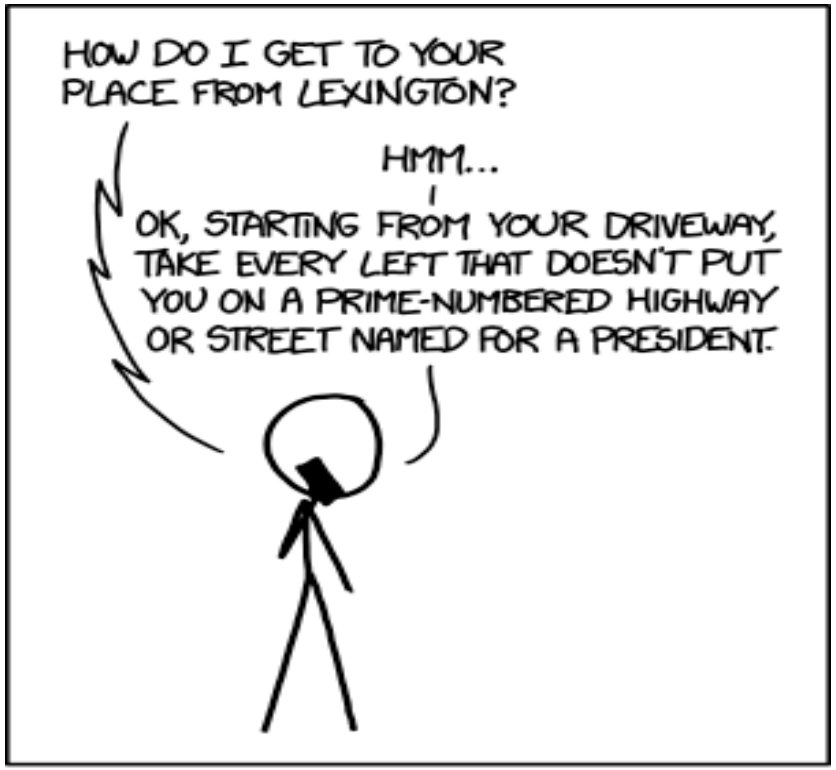
It is also the binary representation of the first decimals of pi.

A short program could generate this random string.

Gell-Mann:
In Evolution “Frozen Accidents” cause regularities

Identify “frozen accidents” (regularities) in genomes

Calculate Effective Complexity as the AIC of the regularities

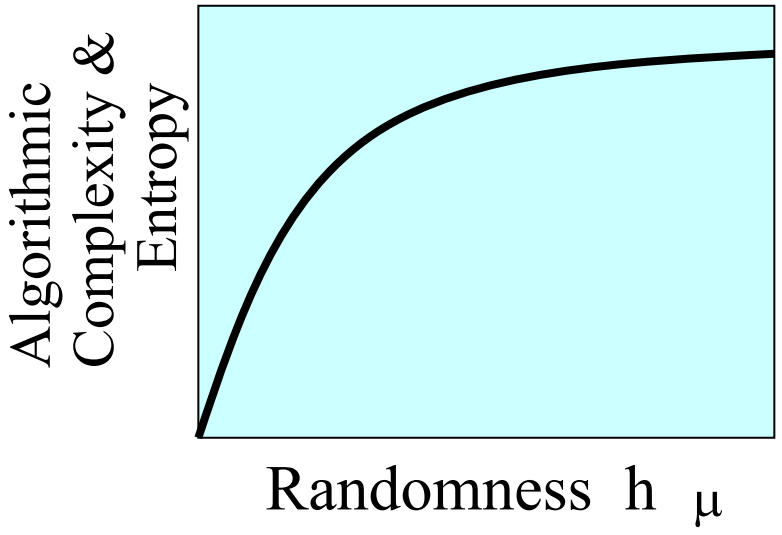


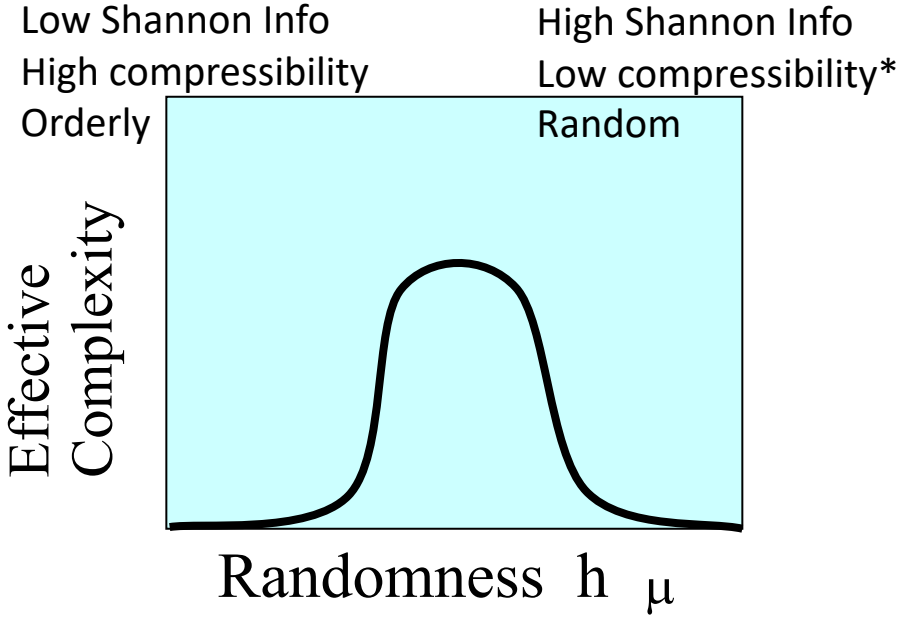
http://c2.com/cgi/wiki?KolmogorovComplexity

Gell-Mann's Effective Complexity

- The length of the shortest description of a set's regularities
- EC(x) = K(r) where r is the set of regularities in x

  and Kolmogorov Complexity (or AIC), K(r), is the length of a concise description of a set
- Highest for entities that are not strictly regular or random





What's a regularity?

- Gell Mann suggests one formal way to identify regularities
- Determine mutual AIC between parts of the string
  - If x = [x1,x2]
  - K(x1,x2) = K(x1) + K(x2) – K (x)
  - The sum of the AICs of the parts – the AIC of the whole
  - Eg. 10010 10011 10010 -the whole has more regularity than the sum of the regularities in the parts
  - Regularities exist when K(x1,x2) > 0

- Identify the regularities, r,
- Calculate K(r), the AIC of the regularities
- Effective complexity = K(r)

Here r = 10010
AIC of 10010 which is O(5)
Effective Complexity is O(5)

Logical Depth

- Bennett 1986;1990:
  - The *Logical depth* of *x* is the run time of the shortest program that will cause a UTM to produce *x* and then halt.
  - Logical depth is not a measure of randomness; it is small both for trivially ordered and random strings.
- Drawbacks:
  - Uncomputable.
  - Loses the ability to distinguish between systems that can be described by computational models less powerful than Turing Machines (e.g., finite-state machines).

- Ay et al 2008, recently proposed proof that strings with high effective complexity also have high logical depth, and low effective complexity have small logical depth.

Total Information

- Alternative approach in Gell-Mann & Lloyd 1998
- EC(x) = K(E) where E is the set of entities of which x is a typical member
- Then K(x) is the length of the shortest program required to specify the members of a the set of which x is a typical member
- Effective complexity measures knowledge—the extent to which the entity is nonrandom and predictable
- Total Information is Effective complexity, K(E), + the Shannon Information of the peculiarities (remaining randomness)
- TI(x) = K(E) + H(x)
- There is a tradeoff between the effective complexity (the completeness of a description of the regularities) and the remaining randomness
- Ex: 10010 10011 10010 10011 10010 10011

Summary of Complexity Measures

- Information-theoretic methods:
  - Shannon Entropy
  - Algorithmic complexity
  - Mutual information
- Effective Complexity:
  - Neither regular nor random entities have high Effective Complexity
- Total Information: Effective Complexity + Entropy
    - AIC of regularities + entropy of what remains

- Computational complexity:
  - How many resources does it take to compute a function?
- The language/machine hierarchy:
  - How complex a machine is needed to compute a function?
- Logical depth:
  - Run-time of the shortest program that generates the phenomena and halts.
- Asymptotic behavior of dynamical systems:
  - Fixed points, limit cycles, chaos

Defining Complexity
Suggested References

- *Computational Complexity* by Papadimitriou. Addison-Wesley (1994).
- *Elements of Information Theory* by Cover and Thomas. Wiley (1991).
- Kaufmann, *At Home in the Universe* (1996) and *Investigations* (2002).
- Per Bak, *How Nature Works: The Science of Self-Organized Criticality* (1988)
- Gell-Mann, *The Quark and the Jaguar* (1994)
- Ay, Muller & Szkola, Effective Complexity and its Relation to Logical Depth, ArXiv (2008)

Kolmogorov Complexity References

- Kolmogorov Complexity With Python
  https://www.youtube.com/watch?v=KyB13PD-UME
- http://en.wikipedia.org/wiki/Kolmogorov_complexity
- http://www.neilconway.org/talks/kolmogorov.pdf
- http://people.cs.uchicago.edu/~fortnow/papers/quaderni.pdf
- http://c2.com/cgi/wiki?KolmogorovComplexity