Topological Reconstruction of a Smooth Manifold-Solid from its Occluding Contour

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Abstract. This paper describes a simple construction for building a combinatorial model of a smooth manifold-solid from a labeled figure representing its occluding contour. The motivation is twofold. First, deriving the combinatorial model is an essential intermediate step in the visual reconstruction of solid-shape from image contours. A description of solid-shape consists of a metric and a topological component. Both are necessary: the metric component specifies how the topological component is embedded in three-dimensional space. The *paneling construction* described in this paper is a procedure for generating the topological component from a labeled figure representing an occluding contour. Second, the existence of this construction establishes the sufficiency of a labeling scheme for line-drawings of smooth solid-objects originally proposed by Huffman[5]. By sufficiency, it is meant that every set of closed plane-curves satisfying this labeling scheme is shown to correspond to a generic view of a manifold-solid. Together with the Whitney theorem[12], this confirms that Huffman's labeling scheme correctly distinguishes possible from impossible solid-objects.

1 Introduction

The larger problem which initiated the research described in this paper is the reconstruction of solid-shape from image contours, a topic at the heart of computer vision. Broadly speaking, it is proposed that the Huffman labeling scheme for smooth solid-objects[5] can function as a two-dimensional intermediate representation, bridging the gap between image contours and three-dimensional solid-objects. While Huffman's influential paper "Impossible Objects as Nonsense Sentences" is widely cited as one source of the Huffman-Clowes junction catalog for trihedral scenes, the last few pages of Huffman's paper is devoted to a labeling scheme for smooth objects (see Figure 1). Huffman's labeling scheme differs from the labeling scheme proposed by Malik[8] in two important ways. First, Malik assumes *piecewise smooth surfaces without boundary*, while Huffman assumes *smooth surfaces with and without boundary*. Second, like other contour labeling work in computer vision (e.g. [3, 6, 11]), Malik considers only visible contours, while Huffman explicitly considers both visible and *occluded* contours.

A smooth surface embedded in three-space generates a set of image contours which can be classified as either: 1) the image of boundaries; or 2) occluding contours. Boundaries are the image of points with neighborhoods topologically equivalent to half-discs (i.e. like the edge of a sheet of paper). Occluding contours are the image of points where the surface is tangent to the viewing direction (i.e. the image of the *contour generator*). Because the surface which forms the boundary of a smooth manifold-solid is closed, its image will contain no boundaries—only occluding contours. Therefore, in this paper, we only consider a subset of the Huffman labeling scheme (i.e. (c) through (f) in Figure 1).

A solid-shape description contains two components. The first is topological, and specifies a set of neighborhoods (i.e. a topology). The second component describes how those neighborhoods are embedded in three-dimensional space. The neighborhoods of the surface which forms the boundary of a manifold-solid can be explicitly represented by means of a combinatorial model called a *paneling.* The term "paneling" is used by Griffiths[4] in his informal but very accessible account of the topology of surfaces. Roughly speaking, a paneling is a set of paper *panels* taped together in prescribed ways. The paneling is produced by applying a straightforward procedure called the *paneling construction* to a labeled figure representing the occluding contours comprising the image of a smooth manifold-solid.

1.1 Solid-Shape from Image Contours

Terzopolous, Kass and Witkin[10] have demonstrated reconstruction of simple solid-shapes from silhouettes under strong assumptions which determine the topology of the solid-shape *a priori.* Basically, they assume that the topology can be described by a tube centered on a user-specified medial-axis. The embedding of the tube in space minimizes an energy functional which combines membrane and thin-plate terms with terms derived from image brightness.

In contrast, it is proposed here that a description of solid-shape can be computed from image contours (or a line-drawing) by means of the three-stage process depicted in Figure 2(a). The first stage is *figural completion*, which is the process of inferring a set of interpolating curves, or *completions*, satisfying the Huffman labeling scheme. The author's recent Ph.D. thesis[13] describes a system which solves figural completion problems in the *anterior scene* domain (i.e. Figure 1(a) and (b)). An anterior scene is a set of smooth surfaces with boundary embedded in three-dimensional space such that the surface normals everywhere have a positive component in the viewing direction. These results, while preliminary, suggest that similiar methods may succeed in the more complex domain of smooth manifold-solids.

The second stage in the reconstruction of a smooth manifold-solid from image contours is the topic of this paper—the paneling construction. The paneling construction translates the labeled figure (which is the product of the figural completion process) into a combinatorial model of an orientable surface without boundary. Stated differently, it makes explicit the topology of the solid-shape implicit in the labeled figure. This is a prerequisite for the final stage—computing a smooth embedding of the paneling in three-dimensional space. For this purpose,

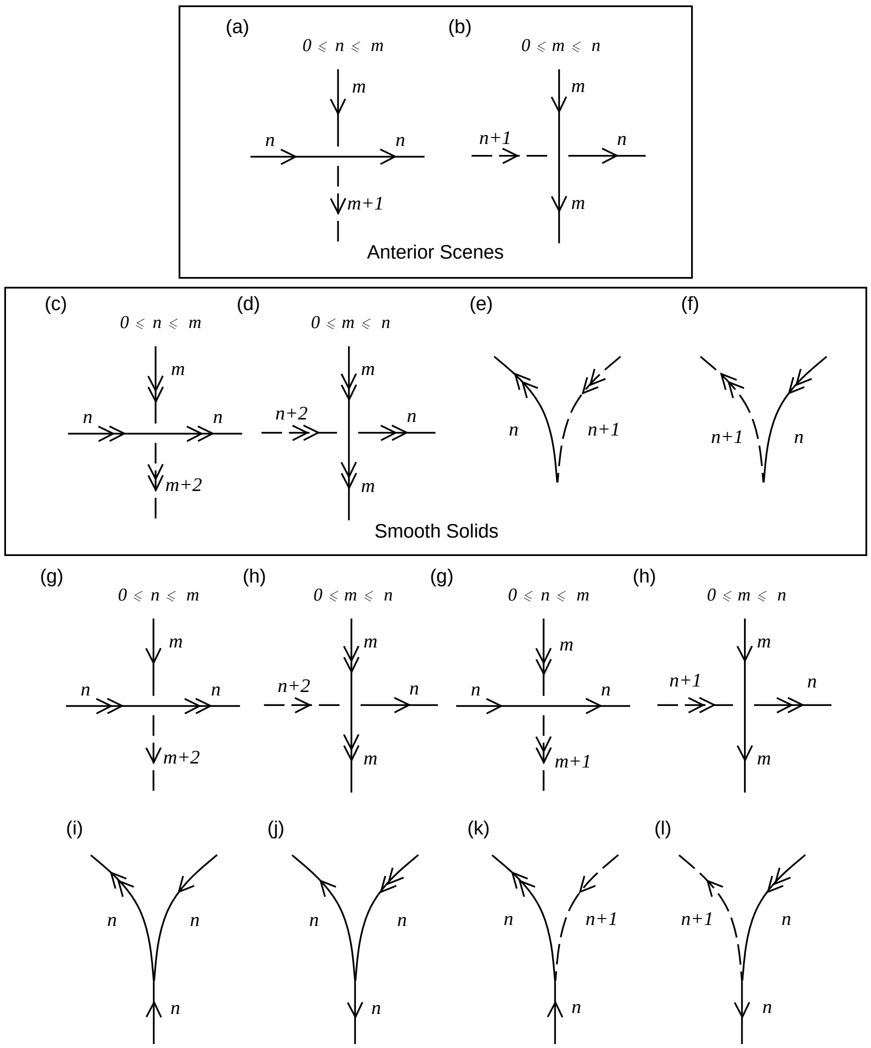

Fig. 1. Huffman's labeling scheme for smooth objects. Single arrows represent boundaries and double arrows represent occluding contours. Numbers are depth indices and direction of arrows indicate sign of occlusion. (a) and (b) Boundary crossing junctions are sufficient to represent the domain of *anterior scenes*[13]. An anterior scene is a set of smooth surfaces with boundary embedded in three-dimensional space such that the surface normals everywhere have a positive component in the viewing direction. (c) through (f) Occluding contour crossing junctions and cusp junctions together define the domain of *smooth manifold-solids*. (g) through (l) Additional junction types required for arbitrary smooth surfaces.

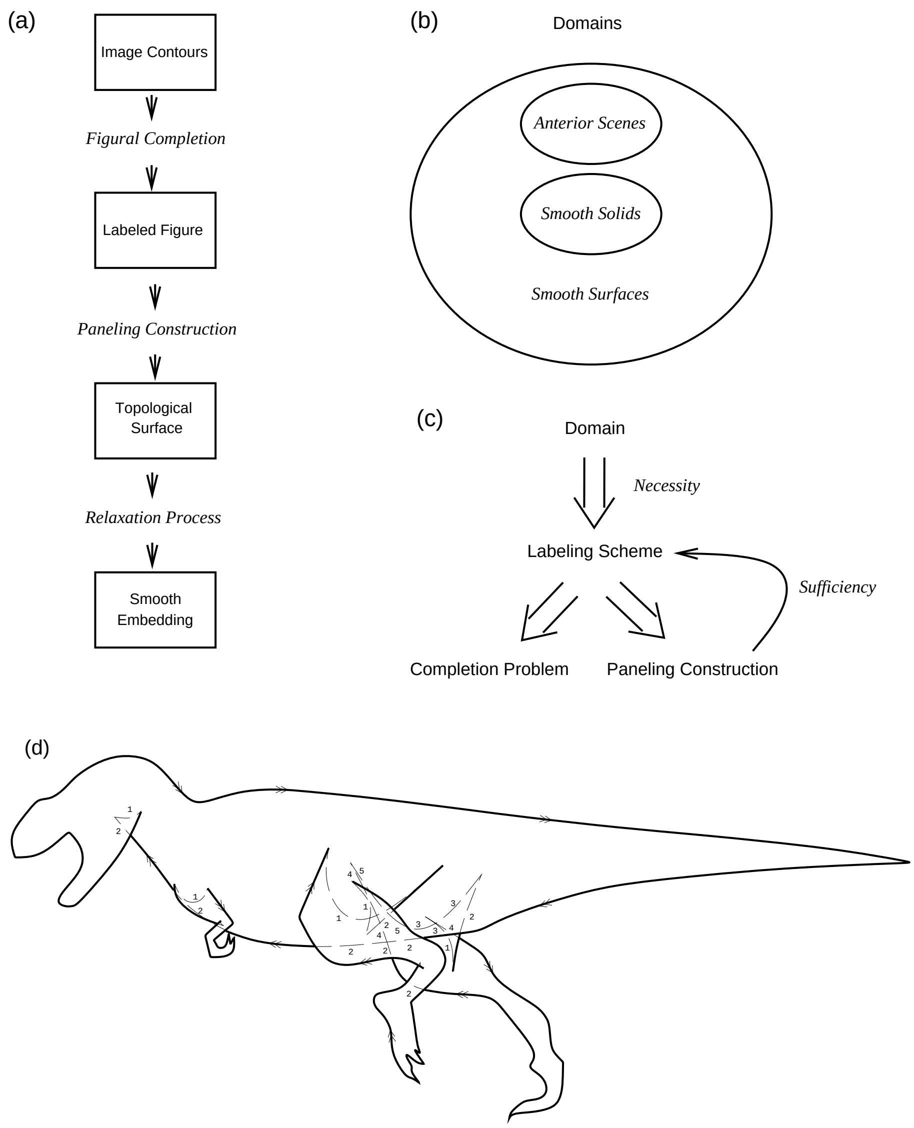

Fig. 2. (a) It is proposed that sets of closed plane-curves satisfying the Huffman labeling scheme can function as a two-dimensional intermediate representation, bridging the gap between image contours and three-dimensional solid-objects. (b) Venn diagram showing relationship of problem domains. (c) For any given problem domain, necessary constraints on depth indices and sign of occlusion define a contour labeling scheme. For each labeling scheme, there is an associated *completion problem* and a *paneling construction*. The existence of a paneling construction for a given labeling scheme establishes its *sufficiency* as a surface representation. (d) Labeled figure representing *Tyrannosaurus rex* solid [Note: Adapted from figure by Gregory S. Paul[9]]. *Figural completion* is the problem of deriving the complete labeled figure from the the visible portion (shown thick).

a relaxation process similiar to that employed by Terzopolous *et al.*[10] would likely suffice.

1.2 Possible and Impossible Smooth Solid-Objects

The other source of motivation for this work is theoretical. The existence of the paneling construction establishes the sufficiency of Huffman's labeling scheme for line-drawings of smooth solid-objects. By sufficiency, it is meant that every set of closed plane-curves satisfying this labeling scheme is shown to correspond to a *generic view* of a smooth manifold-solid. If the view of the manifold-solid is generic, then the crossings will be the only points of multiplicity two in the projection of the contour generator onto the plane:

Defn. *generic view - an image of a smooth manifold-solid where: 1) the multiplicity of the image of the contour generator is one everywhere except at a finite number of points where it is two; and 2) the number of multiplicity two points is invariant to small changes in viewing direction.*

In an influential paper, Koenderink and van Doorn[7] describe the singularities of the visual mapping of a smooth manifold-solid onto the image plane under parallel projection. Largely through this paper, researchers in computer vision have become aware of a theorem due to Whitney which holds that the only generic singularities of mappings of smooth surfaces onto the plane are *folds* and *cusps* (see [12, 1]).

Let $\mathcal{F}$ be the space of figures satisfying the Huffman labeling scheme for smooth solid-objects and let $\mathcal{G}$ be the space of generic views of smooth manifold-solids. Then the Whitney theorem tells us that $\mathcal{G} \subseteq \mathcal{F}$, that is, that there are no generic views of smooth manifold-solids that do not have corresponding labeled figures. In contrast, the paneling construction described in this paper makes it possible to prove the converse (i.e. $\mathcal{F} \subseteq \mathcal{G}$):

Theorem *Every set of closed plane curves satisfying the labeling scheme illustrated below represents a generic view of a manifold-solid.*

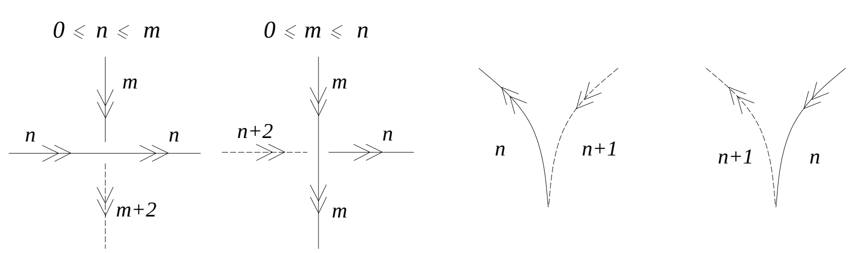

Fig. 3. Huffman labeling scheme for smooth solid-objects.

Together, this confirms that the labeling scheme consisting of (c) through (f) in Figure 1 correctly distinguishes possible from impossible smooth solid-objects (i.e. $\mathcal{G} \equiv \mathcal{F}$).

2 Sufficiency of Labeling Scheme

Unfortunately, it is not possible to present a complete proof of this theorem within the allowed space. Instead, we describe the paneling construction in detail and briefly sketch the remainder of the proof.

Observe that a set of closed plane-curves partitions the plane into regions. The boundary of each planar region is a cycle of oriented edges separated by crossings and cusps. Every edge forms the side of exactly two planar regions, one lying to its right, the other to its left (where right and left are with respect to the edge's orientation). Note that if an edge is the projection of an occluding contour, then the multiplicity of the projection of interior surface points onto image points is two greater on the right side of the edge than on the left. Furthermore, the multiplicity of the projection of interior surface points onto image points will be constant within a planar region.

Let A and B be neighboring regions and let A lie to the right of B. If the labeled figures represents a manifold-solid, and if γ_A and γ_B are the multiplicities of the projection of interior surface points within regions A and B, then $\gamma_A - \gamma_B = 2$. Observe that the set of difference constraints among all neighboring planar regions form the node-edge incidence matrix of a network.

Example

Figure 4(b) illustrates a network constructed in this fashion for the planar partition depicted in Figure 4(a). The linear system of difference equations represented by this network appear below:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \gamma_A \\ \gamma_B \\ \gamma_C \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Recall that a system of difference equations has a solution if and only if the sums of the weights of every cycle in its corresponding network equal zero (where the weight of an edge is k or $-k$ depending on the direction of traversal). In a longer version of this paper[14], we show not only that a solution to this system of difference constraints always exists, but also that a solution exists where the value of γ for every planar region is greater than the largest depth index among all edges bordering that region in the labeled figure. Fortunately, this second condition is easy to satisfy, since it is always the case that if $\{x_1, x_2, ..., x_n\}$ is a solution to a system of difference equations, then $\{x_1 + c, x_2 + c, ..., x_n + c\}$ is also a solution for any constant c. Clearly, a sufficiently large c can always be found, and it is sufficient to prove that the sums of the weights around every closed cycle in a network constructed as described equal zero.

2.1 Paneling Construction

Since each region of the planar partition induced by the labeled figure is a topological disc, flat panels of the same shape and size can be cut out from a sheet of paper. Let us further assume that the paper is white on one side and

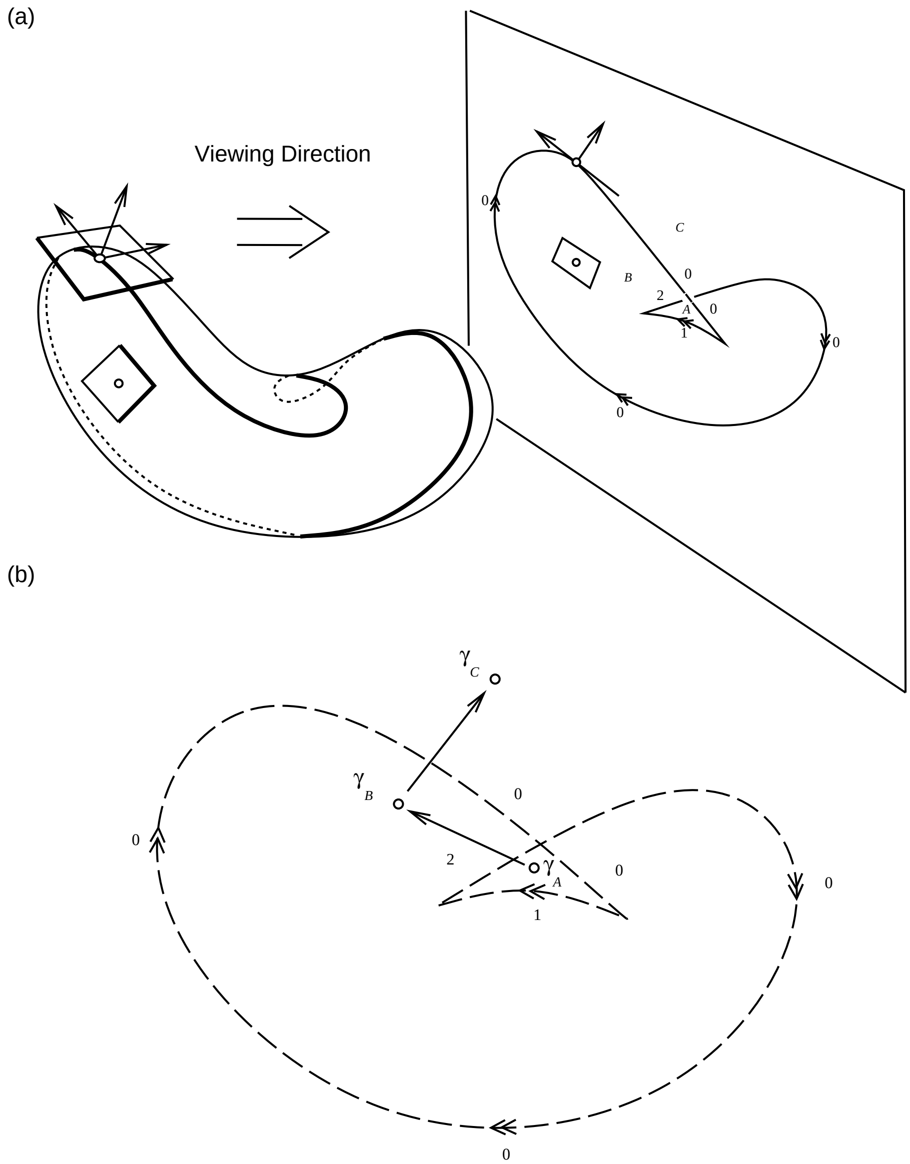

Fig. 4. (a) A kidney bean shaped solid. The *contour generator* is the locus of surface points tangent to the viewing direction (i.e. the pre-image of the occluding contour). Cusps occur when the direction of the contour generator coincides with the viewing direction. Here the occluding contour contains two cusps which form a "swallowtail." [Note: This figure is adapted from a figure by Jim Callahan[2].] (b) The network representing the system of difference equations which must be solved as a precondition for the paneling construction. The labeled figure is shown dashed while network edges are shown solid. Here the solution is $\gamma_A = 4$, $\gamma_B = 2$ and $\gamma_C = 0$.

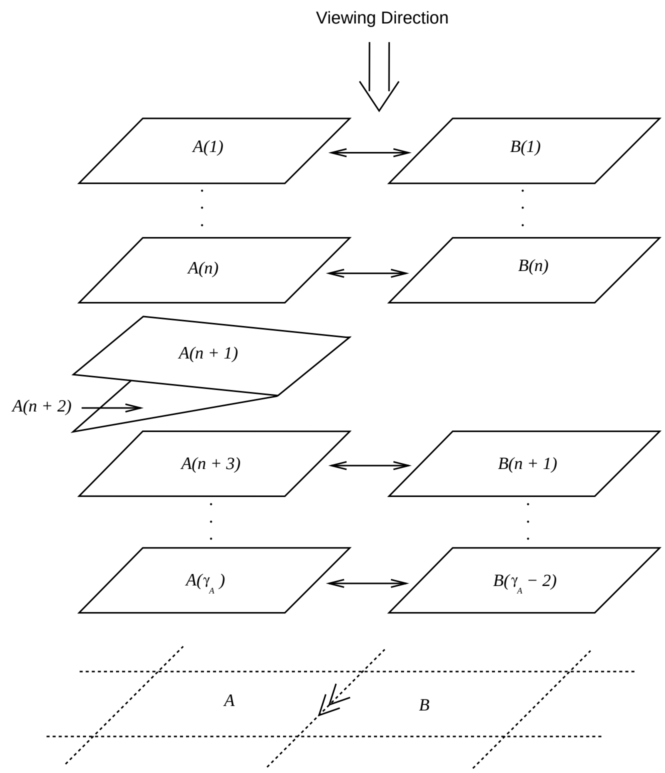

Fig. 5. Paper panels stacked above region A and B in the plane. Following the *identification scheme*, all copies of regions A and B but $A(n+1)$ and $A(n+2)$ are glued along their adjacent sides. Copies $A(n+1)$ and $A(n+2)$ are glued to form a *fold edge*.

black on the other side. For each region, R, create γ_R copies of the paper panel, where γ_R is a solution to the above system of difference equations. Let the copies of region R be $R(1), R(2), ..., R(\gamma_R)$ and let them be arranged in a stack above region R in the plane such that $R(1)$ is the uppermost region and $R(\gamma_R)$ is the lowermost region. This is done so that the white side of each panel faces upward and the black side of each panel faces downward.

Let A and B be neighboring regions and let n be the depth index of the edge separating them. Note that if A lies to the right of B then $\gamma_A - \gamma_B = 2$. Unless n equals zero, identify the side (bordering B) of each panel (above region A) numbered 1 through n with the adjacent side of the corresponding copy of region B such that white is glued to white (i.e. $A(1) \rightleftharpoons B(1), ..., A(n) \rightleftharpoons B(n)$). Then identify the side of $A(n+1)$ (adjacent to B) with the side of $A(n+2)$ (also adjacent to B) such that white and black meet (i.e. $A(n+1) \Rightarrow A(n+2)$). We call an edge where white and black meet a *fold edge*. Now, unless γ_A equals $n+2$, identify the side (bordering B) of each panel (above region A) numbered $n+3$

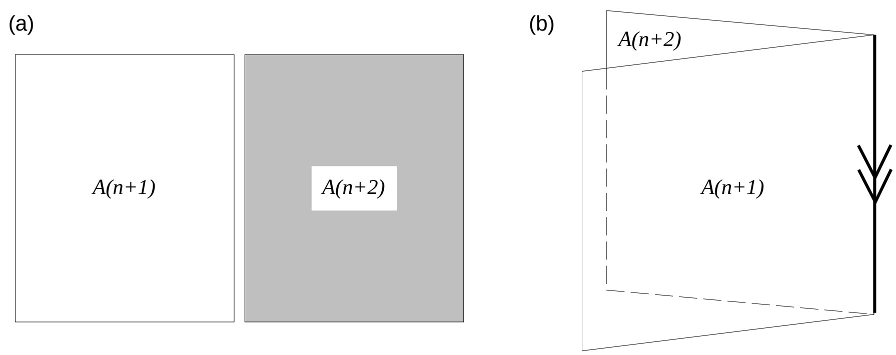

Fig. 6. (a) Flattened view of $A(n+1) \Rightarrow A(n+2)$. (b) The *fold edge* which results.

through γ_A with the adjacent side of the copy of region B numbered $n+1$ through $\gamma_A - 2$ such that white is glued to white (i.e. $A(n+3) \rightleftharpoons B(n+1), ..., A(\gamma_A) \rightleftharpoons B(\gamma_A - 2)$). We refer to this implicitly defined set of edge identifications as the *identification scheme*. The effect of the identification scheme is to create n interior edges above and $\gamma_A - n - 2$ interior edges beneath a fold edge in the paneling. The set of identifications can be divided into three subranges, the first and last of which are potentially empty:

(a) $A(1) \rightleftharpoons B(1), ..., A(n) \rightleftharpoons B(n)$
(b) $A(n+1) \Rightarrow A(n+2)$
(c) $A(n+3) \rightleftharpoons B(n+1), ..., A(\gamma_A) \rightleftharpoons B(\gamma_A - 2)$

By everywhere gluing along the edges specified by the identification scheme, a paneling is created. However, we still must show that this paneling represents a manifold-solid. This can be done by demonstrating that the neighborhood of every point of the paneling resembles an interior surface point (i.e. is homeomorphic to a disc). Towards this end, we observe that points of the paneling can be divided into the following categories: 1) Points interior to a panel; 2) Points lying on a panel edge; 3) Vertices originating in crossings; and 4) Vertices originating in cusps. The first two cases are trivial. First, it is clear that a point interior to a panel forms an interior point of the surface. Second, the nature of the identification scheme ensures that every panel edge is identified with one and only one other edge. Pairs of identified panel edges therefore form interior edges of the paneling.

This leaves only the last two cases (i.e. paneling vertices). These are points where the corners of two or more panels meet and are created when the construction is applied to the edges incident to a crossing or cusp in the labeled figure. In a longer version of this paper[14], we show (by enumeration) that the neigh-

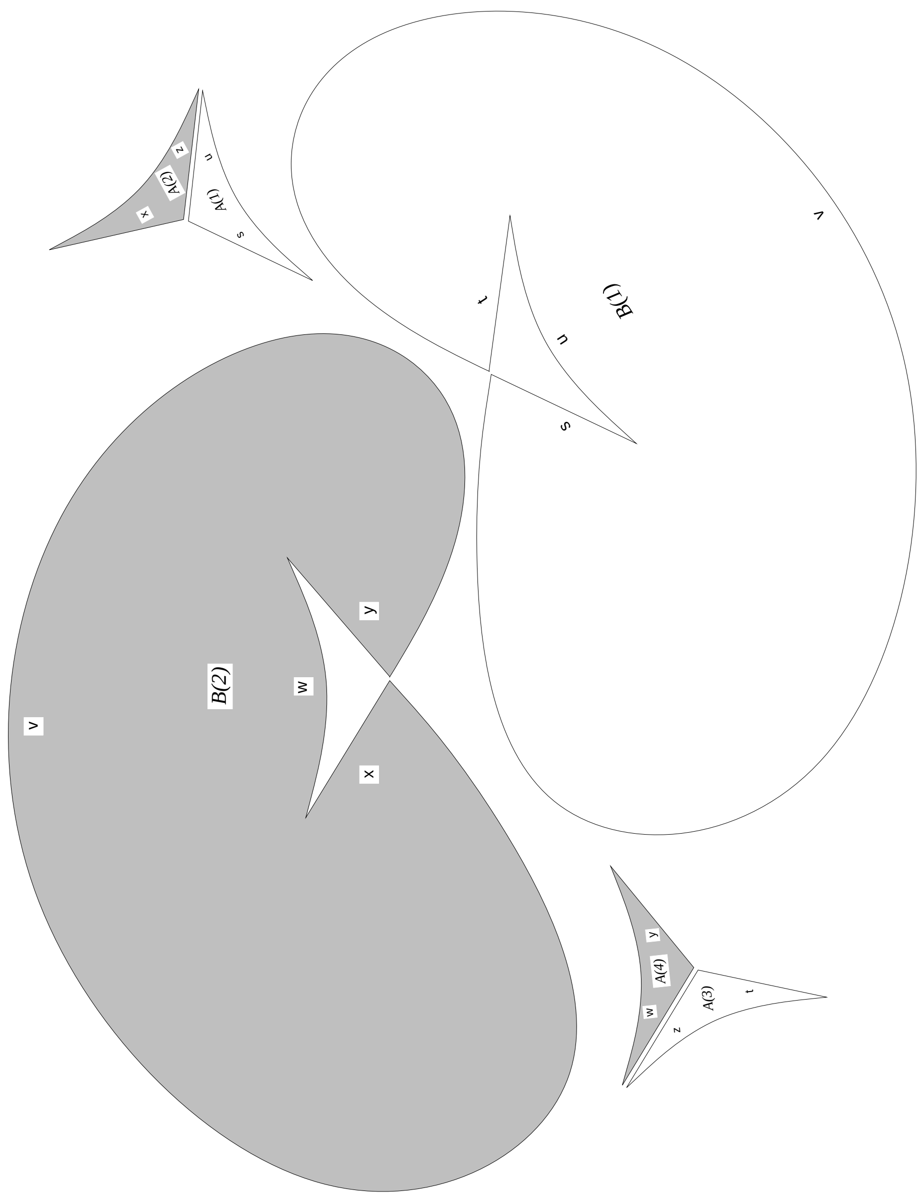

Fig. 7. Paneling produced by the construction for the kidney bean solid. Edges which are adjacent are identified. Additional identifications are indicated by lowercase letters. Construction with scissors and tape yields a paper model of the surface which forms the boundary of the kidney bean.

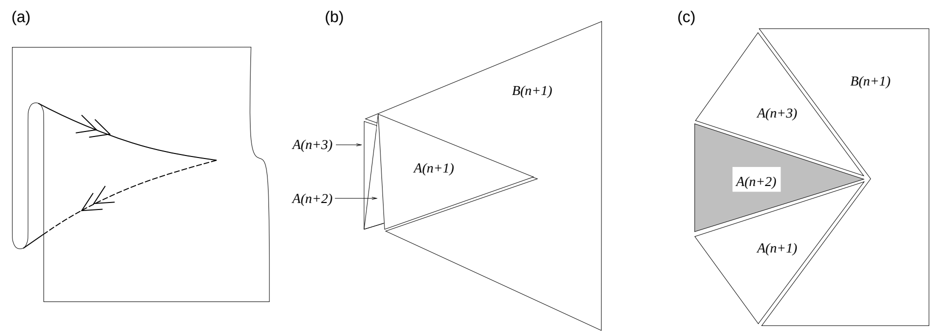

Fig. 8. (a) Tuck in a smooth surface with cusp superimposed. (b) Folded embedding of disc produced by the paneling construction when applied to edges incident at a cusp. (c) Unfolded embedding of disc.

borhoods of all paneling vertices produced by the construction are homemorphic to discs. In lieu of this, in Figure 8, we show how the construction translates a cusp in the labeled figure to a *tuck* in the paneling. By unfolding the tuck, it becomes clear that the neighborhood of the paneling vertex is homeomorphic to a disc.

2.2 All Panelings Form Boundaries of Manifold-Solids

Because the neighborhood of every point of a paneling produced by the construction is homeomorphic to a disc, it follows that all such panelings represent surfaces without boundary. Furthermore, because the construction guarantees that the panelings can be embedded in three-dimensional space without self-intersection, the surfaces without boundary must be orientable. We therefore conclude that all panelings generated by the construction represent the boundaries of manifold-solids.

We now show that the image of the surface without boundary produced by the construction corresponds to the labeled figure in every respect and that the view is generic. First, the definition of the construction guarantees that each edge in the labeled figure produces exactly one fold edge in the paneling. The multiplicity of the projection of the fold is therefore equal to one everywhere except at crossings. Furthermore, at crossings the multiplicity of the projection of the fold is two, since exactly two fold edges are produced in the paneling when the construction is applied to the edges incident at a crossing. It follows that the view is generic. Second, the definition of the construction guarantees that the image of the manifold-solid everywhere lies to the right of the occluding contour so that the sign of occlusion is respected. Finally, the definition of the construction guarantees that the depth of a fold edge everywhere matches the depth index of the labeled figure, since exactly n interior panel edges are assembled above each fold edge.

3 Conclusion

This paper describes a simple construction for building a combinatorial model of a smooth manifold-solid from a labeled figure representing its occluding contour. This is an essential (and previously unaddressed) intermediate step in the reconstruction of solid-shape from image contours. In addition, this paper establishes the sufficiency of the Huffman labeling scheme for smooth solid-objects as a surface representation, and as a source of grouping constraints for image contours.

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