Angel and Shreiner: Interactive Computer Graphics, Sixth Edition

Chapter 3 Odd Solutions

3.1 If the scaling matrix is uniform then

$$\mathbf{RS} = \mathbf{RS}(\alpha, \alpha, \alpha) = \alpha\mathbf{R} = \mathbf{SR}$$

Consider $\mathbf{R}_x(\theta)$, if we multiply and use the standard trigonometric identities for the sine and cosine of the sum of two angles, we find

$$\mathbf{R}_x(\theta)\mathbf{R}_x(\phi) = \mathbf{R}_x(\theta + \phi)$$

By simply multiplying the matrices we find

$$\mathbf{T}(x_1, y_1, z_1)\mathbf{T}(x_2, y_2, z_2) = \mathbf{T}(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

3.5 There are 12 degrees of freedom in the three–dimensional affine transformation. Consider a point $\mathbf{p} = [x, y, z, 1]^T$ that is transformed to $\mathbf{p'} = [x'y', z', 1]^T$ by the matrix $\mathbf{M}$. Hence we have the relationship $\mathbf{p'} = \mathbf{Mp}$ where $\mathbf{M}$ has 12 unknown coefficients but $\mathbf{p}$ and $\mathbf{p'}$ are known. Thus we have 3 equations in 12 unknowns (the fourth equation is simply the identity 1=1). If we have 4 such pairs of points we will have 12 equations in 12 unknowns which could be solved for the elements of $\mathbf{M}$. Thus if we know how a quadrilateral is transformed we can determine the affine transformation.
In two dimensions, there are 6 degrees of freedom in $\mathbf{M}$ but $\mathbf{p}$ and $\mathbf{p'}$ have only x and y components. Hence if we know 3 points both before and after transformation, we will have 6 equations in 6 unknowns and thus in two dimensions if we know how a triangle is transformed we can determine the affine transformation.

3.7 It is easy to show by simply multiplying the matrices that the concatenation of two rotations yields a rotation and that the concatenation of two translations yields a translation. If we look at the product of a rotation and a translation, we find that the left three columns of $\mathbf{RT}$ are the left three columns of $\mathbf{R}$ and the right column of $\mathbf{RT}$ is the right column of the translation matrix. If we now consider $\mathbf{RTR'}$ where $\mathbf{R'}$ is a rotation matrix, the left three columns are exactly the same as the left three columns of $\mathbf{RR'}$ and the and right column still has 1 as its bottom element. Thus, the form is the same as **RT** with an altered rotation (which is the concatenation of the two rotations) and an altered translation. Inductively, we can see that any further concatenations with rotations and translations do not alter this form.

3.9 If we do a translation by -h we convert the problem to reflection about a line passing through the origin. From m we can find an angle by which we can rotate so the line is aligned with either the x or y axis. Now reflect about the x or y axis. Finally we undo the rotation and translation so the sequence is of the form $\mathbf{T}^{-1}\mathbf{R}^{-1}\mathbf{SRT}$.

3.11 The most sensible place to put the shear is second so that the instance transformation becomes $\mathbf{I} = \mathbf{TRHS}$. We can see that this order makes sense if we consider a cube centered at the origin whose sides are aligned with the axes. The scale gives us the desired size and proportions. The shear then converts the right parallelepiped to a general parallelepiped. Finally we can orient this parallelepiped with a rotation and place it where desired with a translation. Note that the order $\mathbf{I} = \mathbf{TRSH}$ will work too.

3.13

$$\mathbf{R} = \mathbf{R}_z(\theta_z)\mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x) =$$

$$\begin{bmatrix} \cos\theta_y\cos\theta_z & \cos\theta_z\sin\theta_x\sin\theta_y - \cos\theta_x\sin\theta_z & \cos\theta_x\cos\theta_z\sin\theta_y + \sin\theta_x\sin\theta_z & 0 \\ \cos\theta_y\sin\theta_z & \cos\theta_x\cos\theta_z + \sin\theta_x\sin\theta_y\sin\theta_z & -\cos\theta_z\sin\theta_x + \cos\theta_x\sin\theta_y\sin\theta_z & 0 \\ -\sin\theta_y & \cos\theta_y\sin\theta_x & \cos\theta_x\cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.17 One test is to use the first three vertices to find the equation of the plane $ax + by + cz + d = 0$. Although there are four coefficients in the equation only three are independent so we can select one arbitrarily or normalize so that $a^2 + b^2 + c^2 = 1$. Then we can successively evaluate $ax + bc + cz + d$ for the other vertices. A vertex will be on the plane if we evaluate to zero. An equivalent test is to form the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$$

for each $i = 4, ...$ If the determinant of this matrix is zero the ith vertex is in the plane determined by the first three.

3.19 Although we will have the same number of degrees of freedom in the objects we produce, the class of objects will be very different. For example if we rotate a square before we apply a nonuniform scale, we will shear the square, something we cannot do if we scale then rotate.

3.21 The vector $a = u \times v$ is orthogonal to u and v. The vector $b = u \times a$ is orthogonal to u and a. Hence, u, a and b form an orthogonal coordinate system.

3.23 Using $r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v}$, with $\theta = 90$ and $\mathbf{v} = (1, 0, 0)$, we find for rotation about the x-axis

$$r = \frac{\sqrt{2}}{2}(1, 1, 0, 0).$$

Likewise, for rotation about the y axis

$$r = \frac{\sqrt{2}}{2}(1, 0, 1, 0).$$

3.29 Expanding repeatedly as in the hint

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \ldots = \alpha_1 P_1 + (\alpha_2 + \alpha_1 - \alpha_1)P_2 + \ldots$$
$$= \alpha_1(P_1 - P_2) + (\alpha_1 + \alpha_2)P_2 + \alpha_3 P_3 + \ldots$$
$$= \alpha_1(P_1 - P_2) + (\alpha_1 + \alpha_2)P_2 + (\alpha_1 + \alpha_2 + \alpha_3 - \alpha_1 - \alpha_2)P_3 + \ldots$$
$$= \alpha_1(P_1 - P_2) + (\alpha_1 + \alpha_2)(P_2 - P_3) + (\alpha_1 + \alpha_2 + \alpha_3)P_3 + \ldots$$

until eventually we have

$$P = \alpha_1(P_1 - P_2) + (\alpha_1 + \alpha_2)(P_2 - P_3) + \ldots +$$

$$(\alpha_1 + \ldots + \alpha_{N-1})(P_N - P_{N-1}) + (\alpha_1 + \alpha_2 + \ldots + \alpha_N)P_N$$

If

$$\alpha_1 + \alpha_2 + \ldots + \alpha_N = 1$$

then, since $1 \times P_N = P_N$, we have the sum of a point and $N - 1$ vectors which yields a point.