Angel: Interactive Computer Graphics, Third Edition

Chapter 4 Solutions

4.1 Consider the homogeneous recurrence

$$y(k) + a_{n-1}y(k-1) + ... + a_0 y(k-n) = 0,$$

We know there are n linearly independent solutions $\{z_i(k)\}i = 1, .., n$ and any solution can be written as

$$y(k) = \sum_{i=0}^{n} c_i z_i(k)$$

where the constants $\{c_i\}$ are determined by the initial conditions. A virtually identical result holds for linear differential equations. If can regard the sequences $\{z_i(k)\}i = 1, .., n$ as basis vectors and thus the solution of the homogeneous equation is a point in the vector space spanned by this basis.
For the inhomogeneous equation

$$y(k) + a_{n-1}y(k-1) + ... + a_0 y(k-n) = x(k),$$

where $\{x(k)\}$ is a known sequence, if $\{u(k)\}$ is *any* solution of this recurrence, any other solution can be written as

$$y(k) = u(k) + \sum_{i=0}^{n} c_i z_i(k).$$

Thus $\{u(k)\}$ acts as a *point* and any solution is another point that is obtained by adding a vector to it.

4.5 There are 12 degrees of freedom in the three–dimensional affine transformation. Suppose we consider a point $\mathbf{p} = [x, y, z, 1]^T$ that is transformed to $\mathbf{p'} = [x'y', z'1]^T$ by the matrix $\mathbf{M}$. Hence we have the relationship $\mathbf{p'} = \mathbf{Mp}$ where $\mathbf{M}$ has 12 unknown coefficients but $\mathbf{p}$ and $\mathbf{p}|'$ are known. Thus we have 3 equations in 12 unknowns (the fourth equation is simply the identity 1=1). If we have 4 such pairs of points we will have 12 equations in 12 unknowns which could be solved for the elements of $\mathbf{M}$. Thus if we know how a quadrilateral is transformed we can determine the affine transformation.

In two dimensions, there are 6 degrees of freedom in $\mathbf{M}$ but $\mathbf{p}$ and $\mathbf{p}'$ have only x and y components. Hence if we know 3 points both before and after transformation, we will have 6 equations in 6 unknowns and thus in two dimensions if we know how a triangle is transformed we can determine the affine transformation.

4.7 It is easy to show by simply multiplying the matrices that the concatenation of two rotations yields a rotation and that the concatenation of two translations yields a translation. If we look at the product of a rotation and a translation, we find left three columns of $\mathbf{RT}$ are the left three columns of $\mathbf{R}$ and the right column of $\mathbf{RT}$ is the right right column of the translation matrix. If we now consider $\mathbf{RTR}'$ where $\mathbf{R}'$ is a rotation matrix, the left three columns are exactly the same as the left three columns of $\mathbf{RR}'$ and the and right column still has 1 as its bottom element. Thus, the form is the same as $\mathbf{RT}$ with an altered rotation (which is the concatenation of the two rotations) and an altered translation. Inductively, we can see that any further concatenations with rotations and translations do no alter this form.

4.9 If we do a translation by -h we convert the problem to reflection about a line passing through the origin. From m we can find an angle by which we can rotate so the line is aligned with either the x or y axis. Now reflect about the x or y axis. Finally we undo the rotation and translation so the sequence is of the form $\mathbf{T}^{-1}\mathbf{R}^{-1}\mathbf{SRT}$.

4.11 The most sensible place to put the shear is second so that the instance transformation becomes $\mathbf{I} = \mathbf{TRHS}$. We can see that this order makes sense if we consider cube centered at the origin whose sides are aligned with the axes. The scale gives us the desired size and proportions. The shear then converts the right parallelepiped to a general parallelepiped. Finally we can orient this parallelepiped with a rotation and place it where desired with a translation. Note that the order $\mathbf{I} = \mathbf{TRSH}$ will work too.

4.13

$$\mathbf{R} = \mathbf{R}_z(\theta_z)\mathbf{R}_y(\theta_y)\mathbf{R}_x(\theta_x) =$$

$$\begin{bmatrix} \cos\theta_y\cos\theta_z & \cos\theta_z\sin\theta_x\sin\theta_y - \cos\theta_x\sin\theta_z & \cos\theta_x\cos\theta_z\sin\theta_y + \sin\theta_x\sin\theta_z & 0 \\ \cos\theta_y\sin\theta_z & \cos\theta_x\cos\theta_z + \sin\theta_x\sin\theta_y\sin\theta_z & -\cos\theta_z\sin\theta_x + \cos\theta_x\sin\theta_y\sin\theta_z & 0 \\ -\sin\theta_y & \cos\theta_y\sin\theta_x & \cos\theta_x\cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.15 It would seem that the matrix $\mathbf{R} = \mathbf{R}_x(45)\mathbf{R}_y(45)$ would be correct but it is not. After the first rotation by 45 degrees, the resulting side view is not symmetric and the required angle of rotation has a cosine of $\frac{\sqrt{6}}{3}$. The angle is approximately 35.36 degrees. We consider this problem in Section 5.3.

4.17 If they are collinear, one vertex is a linear combination of the other two and the determinant of the matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

will be zero.

4.21 The determinant of the matrix is $1+\theta^2$. Repeated multiplications by this matrix increase the determinant so the resulting operation becomes further and further from a rotation matrix causing the point to become further and further from the origin. One remedy is to use the matrix

$$\mathbf{R} = \begin{bmatrix} 1 & -\theta & 0 & 0 \\ \theta & 1-\theta^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which has a determinant of 1.

4.23 Using the notation $\sin\theta_x = s_x$, $\cos\theta_x = c_x$, and likewise for θ_y and θ_z, we find

$$\mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z) = \begin{bmatrix} c_y c_z & -c_y s_z & s_y & 0 \\ -c_z s_x s_y + c_x s_z & -s_x s_y s_z + c_z c_z & -s_x c_y & 0 \\ -c_x s_y c_z + s_x s_z & c_x s_y s_z + s_x c_z & c_x c_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using quaternions, we form the three unit quaternions

$$r_x = \cos\frac{\theta_x}{2} + \sin\frac{\theta_x}{2}(1,0,0),$$
$$r_x = \cos\frac{\theta_y}{2} + \sin\frac{\theta_y}{2}(0,1,0),$$
$$r_x = \cos\frac{\theta_z}{2} + \sin\frac{\theta_z}{2}(0,0,1).$$

We can now compute $r_x r_y r_z$ using quaternion multiplication and obtain a resulting quaternion whose elements correspond to those of the matrix.