Numerical discretization of Monge-Ampere-type equations

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The hedonic model leads to equations which can be described as being of a form of related to the Monge-Ampere equation, e.g.,

$$\rho_1(\nabla\phi(x))\det(H_\phi(x)) = \rho_0(x) \tag{1}$$

where H_ϕ denotes the Hessian of ϕ and ρ_i are given data. Writing $v = \nabla\phi$, we can also express this as

$$\rho_1(v(x))\det(J_v(x)) = \rho_0(x) \tag{2}$$

where J_v denotes the Jacobian of the mapping v.

There has been significant research regarding the existence, uniqueness and regularity of solutions of Monge-Ampere type equations in the past decade (cf. [3] and references in [1]). However, as observed in [1], equations of Monge-Ampere type have "not received much attention from numerical analysts." There are however several different approaches that have been suggested to solve equations of the type (1–2).

One approach [5] involves approximation of the solution ϕ to (1) by piecewise linear functions. The interpretation of the determinant of the Hessian of such weak approximating functions is done geometrically. In short, if ϕ_h is a piecewise linear functions on a triangulation parametrized by h, then $\det(H_\phi(x))$ is evaluated as a sum of point masses at the vertices of the triangulation with weights determined by the geometry of the facets meeting at

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that point. While this approach is appealing geometrically, it is not immediately obvious how to generalize it to more complex equations or to obtain more classical, higher-order approximations following this lead.

A method based on an idea of Moser [6], and further developed in [2] and [1], poses the problem in terms of a flow that transfers from one density to another. We can think of this as a "homotopy method" [7] from a numerical point of view. In the terms of the simple model (1), the original density is ρ_1, and the transformed density is ρ_0. The form of the method given in [2] is as follows.

Let ρ_t denote the linear homotopy between ρ_1 and ρ_0, $t \in [0,1]$:

$$\rho_t(x) = (1-t)\rho_0(x) + t\rho_1(x) \tag{3}$$

We seek $v(x) = v(x,1)$ where $v(x,t)$ satisfies

$$\begin{aligned} \frac{\partial v}{\partial t}(x,t) =& F(v(x,t),t) \quad \forall t \in [0,1] \\ v(x,0) =& x \quad \forall x \end{aligned} \tag{4}$$

where the vector field $F(x,t)$ can be taken [2] to be

$$F(x,t) = \rho_t(x)^{-1} u(x) \tag{5}$$

where

$$\nabla \cdot u(x) = \rho_0(x) - \rho_1(x). \tag{6}$$

The validity of the representaion above can be demonstrated as follows.

Observe that (4) implies that

$$\frac{\partial \det(J_v)}{\partial t}(x,t) = (\nabla \cdot F(v(x,t),t)) \det(J_v)(x,t) \quad \forall t \in [0,1]. \tag{7}$$

Thus it follows that

$$\rho_t(v(x,t)) \det(J_v(x,t)) = \rho_0(x) \quad \forall t \in [0,1] \tag{8}$$

by differentiating this expression with respect to t and noting that (a) the derivative vanishes and (b) the equality holds at $t = 0$.

As observed in [1], the choice of homotopy is not unique, so different approaches are possible. Similarly, the divergence equation (6) is underdetermined, so there are different ways to augment this equation. For example, one could solve the Stokes equations [4].

It is also possible to give a conventional Galerkin formulation for (2), as follows. For simplicity, we take the case where $\rho_1 \equiv 1$. Define a variational form

$$a(v, \phi; w) = \int_\Omega (v_{1,1} w_{2,2} + v_{2,2} w_{1,1} - 2 v_{1,2} w_{1,2})\, \phi \, dx \tag{9}$$

Then consider the variational problem to find v such that

$$a(v, \phi; v) = \int_\Omega \rho_0 \phi \, dx \tag{10}$$

subject to the constraint that v must be a gradient. This constraint could be enforced by a variety of techniques using mixed methods.

Of course, the variational problem (10) is nonlinear, so techniques would have to be developed to resolve the nonlinearity, and homotopy methods might be appropriate. So the method of Moser appears attractive in comparison. However, it is not immediately obvious how far the Moser method can be generalized to more complex equations than the type given in (2).

References

[1] J.-D. Benamou, Y. Brenier. A computational Fluid Mechanics solution to the Monge-Kantorovich mass transfer problem. Numer. Math., 84 (2000), no 3, 375–393.

[2] O. Bokanowski, B. Grébert. Deformations of density functions in molecular quantum chemistry, J. Math. Phys. 37 (1996), no. 4, 1553–1573.

[3] L.C. Evans. Partial differential equations and Monge-Kantorovich mass transfer, Lecture Notes, 2001.

[4] H. Juarez, L. R. Scott, R. Metcalfe, and B. Bagheri. Direct simulation of freely rotating cylinders in viscous flows by high–order finite element methods. *Computers & Fluids*, 29:547–582, 2000.

[5] V. Oliker , L. Prussner. On the numerical solution of the equation $\frac{\partial^2 z}{\partial^2 x} \frac{\partial^2 z}{\partial^2 y} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = f$ and its discretizations, I Numerische Mathematik 54 (1988), 271-293.

[6] J. Moser. On the volume elements on a manifold. Trans. Amer. Math. Soc. 120 (1965) 286–294.

[7] L. T. Watson and L. R. Scott. Solving Galerkin approximations to nonlinear two–point boundary value problems by a globally convergent homotopy method. *SIAM J. Sci. Stat. Comp.*, 8:768–789, 1987.