Constructing Sylvester-type resultant matrices using the Dixon formulation

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Received 14 October 2002; accepted 26 November 2003

Abstract

A new method for constructing Sylvester-type resultant matrices for multivariate elimination is proposed. Unlike sparse resultant constructions discussed recently in the literature or the Macaulay resultant construction, the proposed method does not explicitly use the support of a polynomial system in the construction. Instead, a multiplier set for each polynomial is obtained from the Dixon resultant formulation using an arbitrary term (or a polynomial) for the construction. As shown in the Proceedings of the ACM Symposium on Theory of Computing (1996), the generalized Dixon resultant formulation implicitly exploits the sparse structure of the polynomial system. As a result, the proposed construction for Sylvester-type resultant matrices is \textit{sparse} in the sense that the matrix size is determined by the support structure of the polynomial system, instead of the total degree of the polynomial system.

The proposed construction is a generalization of a related construction proposed by the authors in which the monomial 1 is used (RCWA’ 00, Proceedings of the 7th Rhine Workshop (2000), 167). It is shown that any polynomial (with support inside or outside the support of the polynomial system) can be used instead insofar as that polynomial does not vanish on any of the common zeros of the polynomial system. For generic unmixed polynomial systems (in which every polynomial in the polynomial system has the same support, i.e., the same set of terms), it is shown that the choice of a polynomial does not affect the matrix size insofar as the terms in the polynomial also appear in the polynomial system.

The main advantage of the proposed construction is for mixed polynomial systems. Supports of a mixed polynomial system can be translated so as to have a maximal overlap, and a polynomial is selected with support from the overlapped subset of translated supports. Determining an appropriate translation vector for each support and a term from the overlapped support can be formulated as an optimization problem. It is shown that under certain conditions on the supports of polynomials in

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doi:10.1016/j.jsc.2003.11.003
a mixed polynomial system, a polynomial can be selected leading to a Dixon dialytic matrix of the smallest size, thus implying that the projection operator computed using the proposed construction is either the resultant or has an extraneous factor of minimal degree.

The proposed construction is compared theoretically and empirically, on a number of examples, with other methods for generating Sylvester-type resultant matrices.

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Keywords: Resultant; Cayley–Dixon construction; Dixon method; Dixon resultant formulation; Bézoutians; Sylvester-type matrices; Dialytic method; Dialytic matrices; BKK bound; Support

1. Introduction

Resultant matrices based on the Dixon formulation have turned out to be quite efficient in practice for simultaneously eliminating many variables on a variety of examples from different application domains; for details and comparison with other resultant formulations and elimination methods, see Kapur and Saxena (1995), Chtcherba and Kapur (submitted for publication), Chtcherba (2003) and http://www.cs.panam.edu/~cherba. Necessary conditions can be derived on parameters in a problem formulation under which the associated polynomial system has a solution.

Sylvester-type dialytic matrices based on the Dixon formulation are introduced using a general construction which turns out to be effective especially for mixed polynomial systems. This construction generalizes a construction discussed in our previous work (Chtcherba and Kapur, 2000b). Multiplier sets for each polynomial in a given polynomial system are computed, generating a matrix whose determinant (or the determinant of a maximal minor) includes the resultant. Unlike other Sylvester-type matrix constructions which explicitly use the support of a polynomial system, the proposed construction uses the support only implicitly insofar as the Dixon formulation implicitly exploits the sparse support structure of a polynomial system as proved in Kapur and Saxena (1996). The proposed construction for Sylvester-type resultant matrices is thus sparse in the sense that the matrix size is determined by the support structure of the polynomial system, instead of the total degree of the polynomial system.

It is shown that an arbitrary polynomial can be used to do the proposed construction; the only requirement is that the polynomial does not vanish on any of the common zeros of the polynomial system. For the generic unmixed case (in which each polynomial in the polynomial system has the same support, i.e., the same set of terms), this construction is shown to be optimal if the polynomial used has support contained in the support of the polynomial system. To be precise, given a generic unmixed polynomial system, if the Dixon formulation produces a Dixon matrix whose determinant is the resultant, then the Sylvester-type dialytic matrices (henceforth, called the Dixon dialytic matrices) based on the Cayley–Dixon construction also have the resultant as their determinants. In the case where the Dixon matrix is such that the determinant of the maximal minor has an extraneous factor besides the resultant, the Dixon dialytic matrix does not have an extraneous factor of higher degree. Thus, no additional extraneous factor is contributed to the result by the proposed construction.
For mixed polynomial systems, the proposed construction works especially well. Conditions are identified on the supports of the polynomials in a mixed polynomial system which enable the selection of a term used in the construction such that the resulting Dixon dialytic matrix is of the smallest size. The projection operator computed from this matrix is either the resultant or has an extraneous factor of minimal degree. Heuristics are developed for selecting an appropriate monomial for the construction in the case of mixed polynomial systems which do not satisfy such conditions. Supports are first translated so that they have maximal overlap, providing a large choice of possible terms to be used for selecting polynomial parameters. Determining the translation and selecting a polynomial parameter from the translated supports are formulated as an optimization problem.

The main advantage of using the Dixon dialytic matrices, over the associated Dixon matrices, is (i) in the mixed case, the Dixon dialytic matrices can have resultants as their determinants, whereas the Dixon matrices may have determinants which include, along with the resultants, extraneous factors; (ii) if the determinant of a Dixon dialytic matrix has an extraneous factor along with the resultant, the degree of the extraneous factor is lower than the degree of the extraneous factor appearing in the determinant of the Dixon matrix; (iii) the Dixon dialytic matrices can be stored and computed more efficiently, given that the entries are either zero or the coefficients of the monomials in the polynomials; this is in contrast to the case for the entries of the Dixon matrices, which are determinants in the coefficients.

The next section discusses preliminaries and background—the concept of a multivariate resultant of a polynomial system, the support of a polynomial, the degree of the resultant as determined by the bound developed in a series of papers by Kouchnirenko, Bernstein and Khovanski (also popularly known as the BKK bound), based on the mixed volume of the Newton polytopes of the supports of the polynomials in a polynomial system, Sylvester-type resultant matrices. Section 3 is a review of the generalized Dixon formulation; the Dixon polynomial and Dixon matrix are defined; using the Cauchy–Binet expansion of determinants, the Dixon polynomial and its support are related to the support of the polynomials in the polynomial system.

Section 4 gives the construction for Sylvester-type resultant matrices using the Dixon formulation. Theorem 4.1 serves as the basis of this construction. As the reader will notice, this construction uses an arbitrary polynomial, instead of a construction in Chtcherba and Kapur (2000b) where the particular monomial 1 was used; the only requirement on the selected polynomial is that it should not vanish on any of the common zeros of the polynomial system. In the case where a Dixon dialytic matrix is singular, it is shown how a maximal minor of the matrix can be used for computing the projection operator. It is proved that whenever the Dixon matrix obtained from the generalized Dixon formulation can be used to compute the resultant exactly (up to a sign), the Dixon dialytic matrix can also be used to compute the resultant exactly.

Section 5 discusses how an appropriate polynomial (often selected to be single monomial) can be chosen for the construction so as to minimize the Dixon dialytic matrix for a given polynomial system and, consequently, the degree of the extraneous factor. For unmixed polynomial systems, it is shown that choosing any polynomial with the support in the support of the polynomial system will lead to the Dixon dialytic matrices of the same size. The heuristic for selecting a polynomial parameter for constructing the Dixon dialytic
matrix is especially effective in the case of mixed systems. It is shown that monomials common to all the polynomials in a given polynomial system are good candidates for use in the construction. Supports of polynomials of a polynomial system can be translated so as to maximize the overlap among them. An example is discussed illustrating why translation of the supports of the polynomials in a polynomial system is crucial for getting Dixon dialytic matrices of smaller size.

The construction is compared theoretically and empirically with other methods for generating sparse resultant matrices, including the subdivision method (Canny and Emiris, 2000) and the incremental method (Emiris and Canny, 1995).

Section 7 discusses an application of the Dixon dialytic construction to multigraded systems. It is proved that the proposed construction generates exact matrices for families of generic unmixed systems including multigraded systems, without any a priori knowledge about the structure of such polynomial systems.

2. Multivariate resultant of a polynomial system

Consider a system of polynomials \( F = \{ f_0, f_1, \ldots, f_d \} \),

\[
    f_0 = \sum_{\alpha \in A_0} c_{0,\alpha} x^{\alpha}, \quad f_1 = \sum_{\alpha \in A_1} c_{1,\alpha} x^{\alpha}, \ldots, \quad f_d = \sum_{\alpha \in A_d} c_{d,\alpha} x^{\alpha},
\]

and for each \( i = 0, \ldots, d \), \( A_i \subset \mathbb{N}^d \) and \( k_i = |A_i| - 1 \), \( x^{\alpha} = x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d} \) where \( (c_{i,\alpha}) \) are parameters. We will denote by \( A = \langle A_0, A_1, \ldots, A_d \rangle \), the support of the polynomial system \( F \).

The goal is to derive a condition on parameters \( (c_{i,\alpha}) \) such that the polynomial system \( F = 0 \) has a solution. One can view this problem as the elimination of variables from the polynomial system. Elimination theory tells us that such a condition exists for a large family of polynomial systems, and is called the resultant of the polynomial system. Since the number of equations is more than the number of variables, in general, for arbitrary values of \( c_{i,\alpha} \), the polynomial system \( F \) does not have any solution. The resultant of the above polynomial system can be defined as follows (Emiris and Mourrain, 1999). Let

\[
    W_U = \{(x, c) \in U \times V \mid f_i(x, c) = 0 \text{ for all } i = 0, 1, \ldots, d\},
\]

where \( U = \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_d} \), \( c = (c_{0,\alpha_0}, \ldots, c_{0,\alpha_{k_0}}, \ldots, c_{d,\alpha_0}, \ldots, c_{d,\alpha_{k_d}}) \), and \( U \) is a projective subvariety of dimension \( d \). \( W_U \) is a projective variety. Consider the following projections of this variety:

\[
    \pi_1: W_U \to U, \quad \pi_2: W_U \to V;
\]

\( \pi_2(W_U) \) is the set of all values of the parameters such that the above system of polynomial equations has a solution. Since \( W_U \) is a projective variety, any projection of it is also a projective variety (see Shafarevich, 1994). Therefore, there exists a set of polynomials defining \( \pi_2(W_U) \). If there is only one such polynomial, then \( \pi_2(W_U) \) is a hypersurface, and its defining equation is called the resultant.
If $U$ is $d$ dimensional, then for generic coefficients, any $d$ equations have a finite number of solutions; consequently $\pi_2(W_U)$ is a hypersurface (see Emiris and Mourrain, 1999 and Buse et al., 2000).

**Definition 2.1.** If variety $\pi_2(W_U)$ is a hypersurface, then its defining equation is called the resultant of $\mathcal{F} = \{f_0, f_1, \ldots, f_d\}$ over $U$, denoted as $R_U(f_0, f_1, \ldots, f_d)$.

In the above definition, the resultant is dependent on the choice of the variety $U$. Different resultant construction methods do not define explicitly the variety $U$. Below, we assume it to be the projective closure of some affine set.

The degree of the resultant is determined by the number of roots that the polynomial system has in a given variety $U$. For simplicity, in this article, we will assume that $U$ contains a projective closure of the embedding of $(\mathbb{C}^*)^d$ or toric variety.

### 2.1. Support and degree of the resultant

The convex hull of the support of a polynomial $f$ is called its Newton polytope, and will be denoted as $\mathcal{N}(f)$. One can relate the Newton polytopes of a polynomial system to the number of its roots.

**Definition 2.2** (Gelfand et al., 1994; Cox et al., 1998). If $Q_1, \ldots, Q_d$ are convex hulls, the mixed volume function $\mu(Q_1, \ldots, Q_d)$ is a unique function which is multilinear with respect to the Minkowski sum and scaling operations, and is defined to have the multilinear property

$$
\mu(Q_1, \ldots, aQ_k + bQ'_k, \ldots, Q_d) = a\mu(Q_1, \ldots, Q_k, \ldots, Q_d) + b\mu(Q_1, \ldots, Q'_k, \ldots, Q_d);
$$

to ensure uniqueness, $\mu(Q, \ldots, Q) = d!\text{Vol}(Q)$, where $\text{Vol}(Q)$ is the Euclidean volume of the polytope $Q$.

**Theorem 2.1** (BKK Bound). Given a polynomial system $\{f_1, \ldots, f_d\}$ in $d$ variables $\{x_1, \ldots, x_d\}$ with the support $\langle A_1, \ldots, A_d \rangle$, the number of roots in $(\mathbb{C}^*)^d$, counting multiplicities, of the polynomial system is either infinite or $\leq \mu(A_1, \ldots, A_d)$; furthermore, the inequality becomes an equality when the coefficients of polynomials in the system satisfy genericity requirements.

Since we are interested in overconstrained polynomial systems, usually consisting of $d + 1$ polynomials in $d$ variables, the BKK bound also tells us the degree of the resultant.

In the resultant, the degree of the coefficients of $f_0$ is, furthermore, equal to the number of common roots that the rest of polynomials have. It is possible to choose any $f_i$ and the resultant expression can be expressed by substituting in $f_i$ the common roots of the remaining polynomial system (Pedersen and Sturmfels, 1993). This implies that the degree of the coefficients of $f_i$ in the resultant equals the number of roots of the remaining set of polynomials. We denote the BKK bound of a $d + 1$ polynomial

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1 The set $(\mathbb{C}^*)^d$ is a $d$-dimensional set where coordinates cannot have zero values, that is $\mathbb{C}^* = \mathbb{C} - \{0\}$. 

system (and call it the $d$-fold mixed volume) by $\langle b_0, b_1, \ldots, b_d \rangle$ as well as $B$, where $b_i = \mu(A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_d)$ and $B = \sum_{i=0}^{d} b_i$.

2.2. Resultant matrices

One way to compute the resultant of a given polynomial system is to construct a matrix with a property that whenever the polynomial system has a solution, such a matrix has a deficient rank, thereby implying that the determinant of any maximal minor is a multiple of the resultant. The BKK bound imposes a lower bound on the size of any such matrix.

A simple way to construct a resultant matrix is to use the dialytic method, i.e., multiply each polynomial with a finite set of monomials, and rewrite the resulting system in matrix form. We call such a matrix the dialytic matrix. This alone, however, does not guarantee that a matrix so constructed is a resultant matrix.

**Definition 2.3.** Given a set of polynomials $\{f_1, \ldots, f_k\}$ in variables $x_1, \ldots, x_d$ and finite monomial sets $X_1, \ldots, X_k$, where $X_i = \{x^\alpha \mid \alpha \in \mathbb{N}^d\}$, write $X_i f_i = \{x^\alpha f_i \mid x^\alpha \in X_i\}$. The matrix representing the polynomial system $X_i f_i$ for all $i = 1, \ldots, k$ can be written as

$$
\begin{pmatrix}
X_1 f_1 \\
X_2 f_2 \\
\vdots \\
X_k f_k
\end{pmatrix} = M \times X = 0,
$$

where $X^T = (x^{\beta_1}, \ldots, x^{\beta_k})$ such that $x^\beta \in X$ if there exists $i$ such that $x^\beta = x^\alpha x^\gamma$ where $x^\alpha \in X_i$ and $x^\gamma \in f_i$. Such matrices will be called the dialytic matrices.

If a given dialytic matrix is non-singular, and its corresponding polynomial system has a solution which does not make $X$ identically zero, then its determinant is a multiple of the resultant. Furthermore, the requirement on the matrix to be non-singular (or even square) can be relaxed, as long as it can be shown that its rank becomes deficient whenever there exists a solution; in such cases, the multiple of the resultant can be extracted from a maximal minor of this matrix.

Note that such matrices are usually quite sparse: matrix entries are either zero or coefficients of the polynomials in the original system. Good examples of resultant dialytic matrices are Sylvester (Sylvester, 1853) for the univariate case, and Macaulay (Macaulay, 1916) as well as Newton sparse matrices of Canny and Emiris (2000) for the multivariate case; they all differ only in the selection of multiplier sets $X_i$.

If the BKK bound of a given polynomial system is $\langle b_0, b_1, \ldots, b_d \rangle$, then $|X_i| \geq b_i$. The matrix size must be at least $B$ (the sum of all the $b_i$’s) for it to be a candidate for being the resultant matrix of the polynomial system.

In the following sections, we show how the Dixon formulation can be used to construct dialytic matrices for the multivariate case. We first give a brief overview of the Dixon formulation, and define the concepts of the Dixon polynomial and the Dixon matrix of

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2 By abuse of notation, for some polynomial $f$, by $x^\alpha \in f$, we mean that $x^\alpha$ appears in (the simplified form of) $f$ with a non-zero coefficient, i.e., $\alpha$ is in the support of $f$. 
a given polynomial system. Expressing the Dixon polynomial using the Cauchy–Binet expansion of determinants of a matrix turns out to be useful for illustrating the dependence of the construction on the support of a given polynomial system.

3. The Dixon matrix

In Dixon (1908), Dixon generalized Cayley’s construction of the Bézout matrix for computing the resultant of two univariate polynomials to the bivariate case. In Kapur et al. (1994), Kapur, Saxena and Yang further generalized this construction to the general multivariate case; the concepts of a Dixon polynomial and a Dixon matrix were introduced as well. Below, the generalized multivariate Dixon formulation for simultaneously eliminating many variables from a polynomial system and computing its resultant is reviewed. More details can be found in Kapur and Saxena (1995) and Saxena (1997).

In contrast to dialytic matrices, the Dixon matrix is dense since its entries are determinants of the coefficients of the polynomials in the original polynomial system. It has the advantage of being an order of magnitude smaller in comparison to a dialytic matrix, which is important as the computation of the determinant of a matrix with symbolic entries is sensitive to its size; see Table 2 in Section 6. The Dixon matrix is constructed through the computation of the Dixon polynomial, which is expressed in matrix form.

Let \( \pi_i(x^{\alpha}) = \pi_1^{\alpha_1} \cdots \pi_i^{\alpha_i} \cdots \pi_d^{\alpha_d} \), where \( i \in \{0, 1, \ldots, d\} \), and the \( \pi_i \)'s are new variables; \( \pi_0(x^{\alpha}) = x^{\alpha} \cdot \pi_i \) is extended to polynomials in a natural way as

\[
\pi_i(f(x_1, \ldots, x_d)) = f(\pi_1, \ldots, \pi_i, x_{i+1}, \ldots, x_d).
\]

**Definition 3.1.** Given a polynomial system \( \mathcal{F} = \{f_0, f_1, \ldots, f_d\} \), where \( \mathcal{F} \subset \mathbb{Q}[c][x_1, \ldots, x_d] \), define its **Dixon polynomial** as

\[
\theta(f_0, \ldots, f_d) = \prod_{i=1}^{d} \frac{1}{x_i - x_i} \begin{vmatrix}
\pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d) \\
\pi_1(f_0) & \pi_1(f_1) & \cdots & \pi_1(f_d) \\
\vdots & \vdots & \ddots & \vdots \\
\pi_d(f_0) & \pi_d(f_1) & \cdots & \pi_d(f_d)
\end{vmatrix}
\]

Hence \( \theta(f_0, f_1, \ldots, f_d) \in \mathbb{Q}[c][x_1, \ldots, x_d, \pi_1, \ldots, \pi_d] \), where \( \pi_1, \pi_2, \ldots, \pi_d \) are new variables.

The order in which original variables in \( x \) are replaced by new variables in \( \pi \) is significant in the sense that the Dixon polynomial computed using two different variable orderings may be different.\(^3\)

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\(^3\) In Chcterba and Kapur (2003) and Chcterba (2003), the impact of different variable orderings on the Dixon polynomial and Dixon matrix as well as the projection operator extracted from the Dixon matrix are discussed. In particular, it is shown that for polynomial systems with almost corner-cut supports, the determinant of the Dixon matrix is precisely the resultant if one variable ordering is used, whereas for other variable orderings, an extraneous factor may be present in the determinant. For multigraded polynomial systems discussed in a later section in this paper, variable orderings violating the block structure of the support can lead to resultant matrices which produce extraneous factors.
Definition 3.2. A Dixon polynomial \( \theta(f_0, f_1, \ldots, f_d) \) can be written in bilinear form as
\[
\theta(f_0, f_1, \ldots, f_d) = \overline{X} \Theta X^T,
\]
where \( \overline{X} = (\overline{x}^{\beta_1}, \ldots, \overline{x}^{\beta_k}) \) and \( X = (x^{a_1}, \ldots, x^{a_l}) \) are row vectors. The \( k \times l \) matrix \( \Theta \) is called the Dixon matrix.

Each entry in \( \Theta \) is a polynomial in the coefficients of the original polynomials in \( F \); moreover its degree in the coefficients of any given polynomial is at most 1. Therefore, the projection operator computed using the Dixon formulation can be of at most of degree \(|X|\) in the coefficients of any single polynomial.

We will relate the support of a given polynomial system \( \mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle \) to the support of its Dixon polynomial.

3.1. Relating size of the Dixon matrix to support of the polynomial system

Given a collection of supports, it is often useful to construct a support which contains a single point from each support in the collection. A special notation is introduced for this purpose.

Definition 3.3. Given a polynomial system support \( \mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle \), let \( \sigma = \langle \sigma_0, \sigma_1, \ldots, \sigma_d \rangle \) such that \( \sigma_i \in A_i \) for \( i = 0, \ldots, d \). Denote this relation as \( \sigma \in \mathcal{A} \); clearly \( \{\sigma_0, \sigma_1, \ldots, \sigma_d\} \subset \mathbb{N}^d \); abusing the notation, \( \sigma \) is also treated as a simplex.

Using the above notation, we can express the support of the Dixon polynomial in terms of a sum of smaller Dixon polynomials.

Theorem 3.1 (Chtcherba and Kapur, 2000a). Let \( F = \{f_0, f_1, \ldots, f_d\} \) be a polynomial system and let \( \mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle \) be the support of \( F \). Then
\[
\theta(f_0, f_1, \ldots, f_d) = \sum_{\sigma \in \mathcal{A}} \sigma(\mathbf{c}) \sigma(\mathbf{x}) = \sum_{\sigma \in \mathcal{A}} \theta_{\sigma},
\]
where \( \theta_{\sigma} = \sigma(\mathbf{c}) \sigma(\mathbf{x}) \) and
\[
\sigma(\mathbf{c}) = \begin{vmatrix}
    c_{0,\sigma_0} & c_{0,\sigma_1} & \cdots & c_{0,\sigma_d} \\
    c_{1,\sigma_0} & c_{1,\sigma_1} & \cdots & c_{1,\sigma_d} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{d,\sigma_0} & c_{d,\sigma_1} & \cdots & c_{d,\sigma_d}
\end{vmatrix}
\]
and
\[
\sigma(\mathbf{x}) = \prod_{i=1}^{d} \frac{1}{x_i - x_i}
\begin{vmatrix}
    \pi_0(x^{a_{\sigma_0}}) & \pi_0(x^{a_{\sigma_1}}) & \cdots & \pi_0(x^{a_{\sigma_d}}) \\
    \pi_1(x^{a_{\sigma_0}}) & \pi_1(x^{a_{\sigma_1}}) & \cdots & \pi_1(x^{a_{\sigma_d}}) \\
    \vdots & \vdots & \ddots & \vdots \\
    \pi_d(x^{a_{\sigma_0}}) & \pi_d(x^{a_{\sigma_1}}) & \cdots & \pi_d(x^{a_{\sigma_d}})
\end{vmatrix}
\]

Proof. Let \( \mathcal{A} = \bigcup_{i=0}^{d} A_i \) and \( \mathcal{A} = \{a_1, \ldots, a_d\} \). Let \( c_{i,a_j} \) be the coefficient of monomial \( x^{a_j} \) in polynomial \( f_i \) for \( a_j \in \mathcal{A} \), if monomial \( x^{a_j} \) does not appear in \( f_i \) then \( c_{i,a_j} = 0 \).
Consider

\[ \theta(f_0, \ldots, f_d) = \prod_{i=1}^{d} \frac{1}{x_i - x_i} \begin{vmatrix} \pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d) \\ \pi_1(f_0) & \pi_1(f_1) & \cdots & \pi_1(f_d) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_d(f_0) & \pi_d(f_1) & \cdots & \pi_d(f_d) \end{vmatrix} \]

Using the Cauchy–Binet expansion (Adkins and Weintraub, 1992) for the determinant of a product of matrices, we can expand the above determinant into a sum of product of determinants:

\[ \det(M \times C), \]

where

\[ M = \begin{pmatrix} \pi_0(x_1^d) & \pi_0(x_2^d) & \cdots & \pi_0(x_{d-1}^d) & \pi_0(x_d^d) \\ \pi_1(x_1^d) & \pi_1(x_2^d) & \cdots & \pi_1(x_{d-1}^d) & \pi_1(x_d^d) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \pi_d(x_1^d) & \pi_d(x_2^d) & \cdots & \pi_d(x_{d-1}^d) & \pi_d(x_d^d) \end{pmatrix} \]

and

\[ C = \begin{pmatrix} c_{0, a_1} & c_{1, a_1} & \cdots & c_{d, a_1} \\ c_{0, a_2} & c_{1, a_2} & \cdots & c_{d, a_2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{0, a_{d-1}} & c_{1, a_{d-1}} & \cdots & c_{d, a_{d-1}} \\ c_{0, a_d} & c_{1, a_d} & \cdots & c_{d, a_d} \end{pmatrix} \]

Using the Cauchy–Binet expansion (Adkins and Weintraub, 1992) for the determinant of a product of matrices, we can expand the above determinant into a sum of product of determinants:

\[ \det(M \times C) = \sum_{1 \leq i_0 < \cdots < i_d \leq d+1} \det(M_{i_0, \ldots, i_d}) \det(C_{i_0, \ldots, i_d}), \]

where \( M_{i_0, \ldots, i_d} \) is a \((d+1) \times (d+1)\) submatrix of \( M \) containing columns \( i_0, i_1, \ldots, i_d \) and \( C_{i_0, \ldots, i_d} \) is a \((d+1) \times (d+1)\) submatrix of \( C \) containing rows \( i_0, i_1, \ldots, i_d \). For multi-index \( [i_0, i_1, \ldots, i_d] \), let \( \sigma = [a_{i_0}, a_{i_1}, \ldots, a_{i_d}] \); then,

\[ \sigma(c) = \det(C_{i_0, \ldots, i_d}) \quad \text{and} \quad \sigma(x) = \prod_{i=1}^{d} \frac{1}{x_i - x_i} \det(M_{i_0, \ldots, i_d}). \]

Note that the order of the \( \sigma_j \)'s in \( \sigma = [\sigma_0, \sigma_1, \ldots, \sigma_d] \) does not change the product \( \sigma(c) \sigma(x) \).
Below we show that it suffices to have the sum over all $\sigma \in A$ where $\sigma_j = a_{ij} \in A_j$ for $j = 0, \ldots, d$.

Given an arbitrary $\sigma$, if is impossible to rearrange $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_d)$ so that each $\sigma_j \in A_j$ for $j = 0, 1, \ldots, d$ (i.e. $\sigma \notin A$); then it is easy to see that $\sigma(\mathbf{c}) = 0$. Assume that for all rearrangements of $\sigma_i$’s in $\sigma$, there is some $\sigma_i \notin A_i$. That means that the entry in the column corresponding to $\sigma_i$ and in the row $i$ (on the diagonal) of the matrix of $\sigma(\mathbf{c})$ is zero. Rearranging $\sigma_i$’s in $\sigma$ amounts to permuting columns of $\sigma(\mathbf{c})$. Since it is impossible to rearrange $\sigma$ so that $\sigma_i \in A_i$, it is impossible to rearrange columns of $\sigma(\mathbf{c})$ so that the diagonal does not have a zero. Consequently, the determinant of $\sigma(\mathbf{c})$ is zero. Hence the expansion

$$\theta(f_0, f_1, \ldots, f_d) = \sum_{\sigma \in A} \sigma(\mathbf{c}) \sigma(\mathbf{x})$$

is a reduced version of the Cauchy–Binet expansion. \square

The above identity shows that if generic coefficients are assumed in the polynomial system, then the support of the Dixon polynomial depends entirely on the support of the polynomial system, as the $\sigma(\mathbf{c})$ would not vanish or cancel each other. To emphasize the dependence of $\theta$ on $A$, the above identity can also be written as $\theta_A = \sum_{\sigma \in A} \theta_{\sigma}$.

We define the support of the Dixon polynomial as

$$\Delta_A = \{ \alpha | x^{\alpha} \in \theta(f_0, f_1, \ldots, f_d) \},$$

where $A = (A_0, A_1, \ldots, A_d)$ and $A_i$ is the support of $f_i$. Let

$$\Delta_A = \{ \beta | \tilde{\mathbf{x}}^{\beta} \in \theta(f_0, f_1, \ldots, f_d) \}.$$ 

For the generic case, using the reduced Cauchy–Binet formula,

$$\Delta_A = \bigcup_{\sigma \in A} \Delta_{\sigma},$$

where $\Delta_{\sigma} = \{ \alpha | x^{\alpha} \in \theta_{\sigma} \}$, because of genericity, $\theta_{\sigma}$ does not cancel any part of $\theta_{\tau}$ for any $\sigma, \tau \in A$ and $\sigma \neq \tau$.

One of the properties of $\sigma(\mathbf{x})$ that we will use is that

$$\Delta_{\sigma} = \{ \alpha | x^{\alpha} \in \theta_{\sigma} |_{\overline{x} = 1} \};$$

that is, substituting $\overline{x}_i = 1$ for $i = 1, \ldots, d$, does not change the support of the Dixon polynomial. This can be seen by noting that given a monomial in the expansion of the determinant of $\sigma(\mathbf{x})$ in terms of variables $x_1, \ldots, x_d$, its coefficient in terms of variables $\overline{x}_1, \ldots, \overline{x}_d$ can be uniquely identified. This is because each monomial in $\theta_{\sigma}$ is of the same degree in terms of variables $x_i, \overline{x}_i$. Hence, substituting $\overline{x}_i = 1$ will not cancel any monomials; if there was cancellation, it should happen without making any substitution.

4. Dixon dialytic matrix

We define a Dixon dialytic matrix which is related to the Dixon matrix in the same way as the Sylvester matrix is related to Bézout’s. In fact the first relationship generalizes the
second. This formulation also generalizes some of the earlier results which first appeared in Chtcherba and Kapur (2000b).

4.1. Construction

Let $g$ be an arbitrary polynomial in variables $\{x_1, x_2, \ldots, x_d\}$ and parameters $(c_{i, \alpha})$. For brevity, let

$$\theta = \theta(f_0, f_1, \ldots, f_d), \quad \text{and also} \quad \theta_i(g) = \theta(f_0, \ldots, f_{i-1}, g, f_{i+1}, \ldots, f_d).$$

Recall that in Chtcherba and Kapur (2000b),

$$\theta_i = \theta_i(1) = \theta(f_0, \ldots, f_{i-1}, 1, f_{i+1}, \ldots, f_d).$$

**Theorem 4.1.**

$$g\theta(f_0, f_1, \ldots, f_d) = \sum_{i=0}^{d} f_i \theta_i(g).$$

**Proof.** Let $a_{i, j}$ and $q_j$ for $i = 0, \ldots, d$ and $j = 1, \ldots, d$ be arbitrary. In general, the following sum of determinants is zero, that is,

$$= \sum_{i=0}^{d} f_i \sum_{j=1}^{d} (-1)^{i+j} q_j$$

$$\begin{vmatrix}
  f_0 & \cdots & f_i-1 & 0 & f_{i+1} & \cdots & f_d \\
  a_{0,1} & \cdots & a_{i-1,1} & q_1 & a_{i+1,1} & \cdots & a_{d,1} \\
  \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{0,d} & \cdots & a_{i-1,d} & q_d & a_{i+1,d} & \cdots & a_{d,d}
\end{vmatrix}$$

$$= \sum_{j=1}^{d} (-1)^{j} q_j \sum_{i=0}^{d} (-1)^{i} f_i$$

$$\begin{vmatrix}
  f_0 & \cdots & f_i-1 & f_{i+1} & \cdots & f_d \\
  a_{0,1} & \cdots & a_{i-1,1} & a_{i+1,1} & \cdots & a_{d,1} \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
  a_{0,j-1} & \cdots & a_{i-1,j-1} & a_{i+1,j-1} & \cdots & a_{d,j-1} \\
  a_{0,j+1} & \cdots & a_{i-1,j+1} & a_{i+1,j+1} & \cdots & a_{d,j+1} \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
  a_{0,d} & \cdots & a_{i-1,d} & a_{i+1,d} & \cdots & a_{d,d}
\end{vmatrix}$$
\[
\begin{vmatrix}
 f_0 & \ldots & f_{i-1} & f_i & f_{i+1} & \ldots & f_d \\
 f_0 & \ldots & f_{i-1} & f_i & f_{i+1} & \ldots & f_d \\
 a_{0,1} & \ldots & a_{i-1,1} & a_{i,1} & a_{i+1,1} & \ldots & a_{d,1} \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 a_{0,j-1} & \ldots & a_{i-1,j-1} & a_{i,j-1} & a_{i+1,j-1} & \ldots & a_{d,j-1} \\
 a_{0,j+1} & \ldots & a_{i-1,j+1} & a_{i,j+1} & a_{i+1,j+1} & \ldots & a_{d,j+1} \\
 \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 a_{0,d} & \ldots & a_{i-1,d} & a_{i,d} & a_{i+1,d} & \ldots & a_{d,d} \\
\end{vmatrix} = 0,
\]

as every determinant in the last sum has the same first two rows.

From the above relation, we can see that

\[
g\theta(f_0, f_1, \ldots, f_d) = \prod_{i=1}^{d} \frac{1}{x_i - x_i} \sum_{i=0}^{d} f_i \begin{vmatrix}
 \pi_0(f_0) & \pi_0(f_1) & \ldots & \pi_0(f_d) \\
 \pi_1(f_0) & \pi_1(f_1) & \ldots & \pi_1(f_d) \\
 \vdots & \vdots & \ddots & \vdots \\
 \pi_d(f_0) & \pi_d(f_1) & \ldots & \pi_d(f_d) \\
\end{vmatrix}
\]

where the last equality is obtained by adding in the sum of Eq. (1) using multilinearity of determinants. Since \( q_i \) can be anything, we choose \( q_i = \pi_i(g) \), and since \( g = \pi_0(g) \),
we have

\[ g^\theta(f_0, f_1, \ldots, f_d) = \sum_{i=0}^{d} f_i \prod_{j=1}^{d} \frac{1}{x_j - x_i} \begin{vmatrix} \pi_0(f_0) & \ldots & \pi_0(f_i-1) & \pi_0(g) & \pi_0(f_{i+1}) & \ldots & \pi_0(f_d) \\ \pi_1(f_0) & \ldots & \pi_1(f_i-1) & \pi_1(g) & \pi_1(f_{i+1}) & \ldots & \pi_1(f_d) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \pi_d(f_0) & \ldots & \pi_d(f_i-1) & \pi_d(g) & \pi_d(f_{i+1}) & \ldots & \pi_d(f_d) \end{vmatrix} \]

\[ = \sum_{i=0}^{d} f_i^\theta(f_0, \ldots, f_i-1, g, f_{i+1}, \ldots, f_d) = \sum_{i=0}^{d} f_i \theta_i(g). \Box \]

In the case where \( g = 1 \), the above identity was already used in Chtcherba and Kapur (2000b) as well Cardinal and Mourrain (1996) to show that the Dixon polynomial is in the ideal of the original polynomial system. As we shall see later, using a general polynomial \( g \) enables us to build smaller Dixon dialytic matrices as there is a choice in selecting the polynomial \( g \).

In bilinear form,

\[ \theta_i(g) = \overline{X_i} \Theta_i(g) X_i, \]

where \( \Theta_i(g) \) is the Dixon matrix of \( \{f_0, \ldots, f_i-1, g, f_{i+1}, \ldots, f_d\} \). Expressing \( \theta_i(g) \) in terms of the \( \Theta_i(g) \) matrix, we have

\[ \theta_i(g) f_i = (\overline{X_i} \Theta_i(g) X_i) f_i = (\overline{X_i} \Theta_i(g)) (X_i f_i). \]

Thus, we can construct a dialytic matrix by using monomial multipliers \( X_i \) for \( f_i \):

\[ M \times Y = \begin{pmatrix} X_0 f_0 \\ X_1 f_0 \\ \vdots \\ X_d f_d \end{pmatrix}. \]

Using the above notation, we can rewrite the formula for the Dixon polynomial,

\[ g^\theta(f_0, f_1, \ldots, f_d) = \overline{X} \Theta g X \]

\[ = \sum_{i=0}^{d} \theta_i(g) f_i \quad \text{by Theorem 4.1} \]

\[ = \sum_{i=0}^{d} \overline{X_i} \Theta_i(g) (X_i f_i) \]

\[ = \overline{Y} (\Theta_0'(g) : \Theta_1'(g) : \cdots : \Theta_d'(g)) \begin{pmatrix} X_0 f_0 \\ X_1 f_1 \\ \vdots \\ X_d f_d \end{pmatrix} \]

\[ = \overline{Y} (T \times M) Y = \overline{Y} \Theta' Y, \]
where \( Y = \bigcup_{i=0}^{d} X_i \), matrix \( \Theta' = T \times M \) and \( \Theta'_i(g) \) is \( \Theta_i(g) \) with possibly extra zero rows, that is \( Y \Theta'_i(g) X_i = X_i \Theta_i(g) X_i \) since \( X_i \subseteq Y \). Therefore,
\[
\overline{X} \Theta' g X = \overline{Y} \Theta' Y.
\]
Note that \( g X \subseteq Y \), and \( X \subseteq Y \); therefore, \( \Theta \) and \( \Theta' \) are the same matrices except for \( \Theta' \) having some extra zero rows and/or columns.

**Corollary 4.1.1.** Given a polynomial system \( \mathcal{F} = \{f_0, \ldots, f_d\} \), its Dixon matrix can be factored as a product of two matrices one of which is a dialytic matrix. That is
\[
\Theta = T \times M,
\]
where \( M \) is the Dixon dialytic matrix of the polynomial system \( \mathcal{F} \).

### 4.2. Univariate case: Sylvester and Dixon dialytic matrices

Let
\[
\begin{align*}
    f_0 &= a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1} + a_m x^m, \\
    f_1 &= b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1} + b_n x^n.
\end{align*}
\]
For the polynomial system \( \{f_0, f_1\} \subset \mathbb{C}[x] \), the Dixon formulation is better known as Bézoutian after Bézout who gave this construction for univariate polynomials. Here in the expression \( \Theta = T \times M \), we will note that \( M \) is the well known Sylvester matrix and \( \Theta \) is the Bézout matrix. The Sylvester matrix for \( \{f_0, f_1\} \) is given by
\[
M = \begin{bmatrix}
    a_0 & a_1 & \cdots & a_m \\
    a_0 & a_1 & \cdots & a_m \\
    \vdots & \vdots & \ddots & \vdots \\
    b_0 & b_1 & \cdots & b_n \\
    b_0 & b_1 & \cdots & b_n \\
\end{bmatrix},
\]
and \( T = (\Theta_0 : \Theta_1) \) where \( \theta_0 = \theta(1, f_1) = \overline{X}_0 \Theta_0 X_0 \) and \( \theta_1 = \theta(f_0, 1) = \overline{X}_1 \Theta_1 X_1 \). For instance,
\[
\theta(1, f_1) = \frac{1}{x - x} \left| \begin{array}{cc}
1 & f_1(x) \\
1 & f_1(x) \end{array} \right| = \frac{f_1(x) - f_1(x)}{x - x} = b_1 S_1 + b_2 S_2 + \cdots + b_n S_n,
\]
where \( S_k = \sum_{i=1}^{k} x^{i-1} x^{k-i} \).

Note that \( X_0 = \{1, x, \ldots, x^{n-1}\} \) and \( X_1 = \{1, x, \ldots, x^{m-1}\} \) as in the Sylvester matrix construction. From this, it can be seen that the \( \Theta_0 \) matrix is \( n \times n \), and hence, similarly, \( \Theta_1 \) will be \( m \times m \), and the matrix \( T \) is of size \( \max\{m, n\} \times (n + m) \). For the univariate case
we can write down the matrix $T$ as

$$T = \begin{pmatrix}
  a_m & b_n & a_m & a_{m-1} \\
  b_n & b_{n-1} & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \ddots \\
  b_n & b_{n-1} & b_2 & a_m & a_{m-1} & a_2 & a_1 \\
  b_n & b_{n-1} & b_2 & b_1 & a_m & a_{m-1} & a_2 & a_1
\end{pmatrix}.$$  

It can be seen from the above construction that when $m > n$, $\det(\Theta)$, the determinant of the Dixon matrix, has the extra factor of $a_m^{m-n}$, which comes from the matrix $T$. This can be verified using the Cauchy–Binet formula for the determinant of a product of non-square matrices (Adkins and Weintraub, 1992).

4.3. Example: bivariate case

Let

$$f_0 = a_{00} + a_{01}y + a_{10}x + a_{11}xy + a_{02}y^2 + a_{20}x^2,$$
$$f_1 = b_{00} + b_{01}y + b_{10}x + b_{11}xy + b_{02}y^2 + b_{20}x^2,$$
$$f_2 = c_{00} + c_{01}y + c_{10}x + c_{11}xy + c_{02}y^2 + c_{20}x^2.$$  

In the expression $\Theta = T \times M$, matrices $T$ and $M$ can be split up into three blocks:

$$T = (\Theta_0 \mid \Theta_1 \mid \Theta_2), \quad M = \begin{pmatrix}
  X_0f_0 \\
  X_1f_1 \\
  X_2f_2
\end{pmatrix}.$$  

Since, in the unmixed case, $X_0 = X_1 = X_2 = \{y^2, xy, x, y, 1\}$, the structure of $M$ is simple to see: it has 15 rows and 14 columns. Also, the monomial structure of $\theta_i$ is the same for $i = 0, 1, 2$, and, hence, matrices $\Theta_i$ have the same layout. We illustrate the matrix $\Theta_0$; the other matrices are similar:

$$\Theta_0 = \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 & [02.20] \\
  0 & 0 & 0 & 0 & [20.11] \\
\end{pmatrix},$$

where $[ij,kl] = \begin{pmatrix}
  b_{ij} & b_{kl} \\
  c_{ij} & c_{kl}
\end{pmatrix}$.  

The above relationship between the Dixon dialytic matrices and the Dixon matrices has also been studied in Zhang (2000) for the bivariate bi-degree case. The construction discussed above in Section 4.1 is general in the sense that it works for any number of variables and any degree. For the bivariate bi-degree case, the above construction reduces to the construction given in Zhang (2000).
4.4. Maximal minors

It was shown in Kapur et al. (1994) that under certain conditions, any maximal minor of the Dixon matrix is a projection operator (i.e., a non-trivial multiple of the resultant). Buse et al. (2000) has derived that any maximal minor of the Dixon matrix is a projection operator of a certain variety which is the projective closure of the associated parametrized affine set. These results immediately apply to the Dixon dialytic matrix, establishing that it is a resultant matrix.

**Theorem 4.2 (Kapur et al., 1994; Saxena, 1997).** The determinant of a maximal minor of the Dixon matrix \( \Theta \) of a polynomial system \( \mathcal{F} \) is a projection operator, that is

\[
\det(\text{minor}_{\text{max}}(\Theta)) = e \mathcal{R}_U(\mathcal{F}),
\]

where \( \mathcal{R}_U(\mathcal{F}) \) is the resultant of the polynomial system \( \mathcal{F} \) over the associated variety \( U \),\(^4\) that is, \( \mathcal{F}(x, v) \equiv 0 \) for \( v \in V \) has a solution in \( U \) if and only if \( \mathcal{R}_U(v) = 0 \).

In general, the underlying resultant variety is not explicit, but usually it can be shown to “contain” an open subset of the affine space (Buse et al., 2000); hence the resultant is a necessary and sufficient condition for the polynomial system to have a solution in that open subset.

The next theorem shows that the resultant is a factor in the projection operators computed from the Dixon dialytic matrix using maximal minor construction. Note that in the construction of a Dixon dialytic matrix, we must assume that for \( v \in V \) such that \( \mathcal{F}(x, v) \equiv 0 \) has a solution in the respective variety, the polynomial parameter \( g \) used to construct the Dixon dialytic matrix does not vanish on that solution. If the variety being considered is the projective closure of \((\mathbb{C}^*)^d \),\(^5\) any monomial chosen for the construction satisfies this condition.

**Theorem 4.3.** Given a polynomial system \( \mathcal{F} = \{f_0, f_1, \ldots, f_d\} \), if, for its Dixon matrix \( \Theta \),

\[
G_{\Theta} = e_{\Theta} \mathcal{R}_U, \quad \text{where } G_{\Theta} = \gcd(\det(\text{minor}_{\text{max}}(\Theta))),
\]

then, for the Dixon dialytic matrix \( M \) of \( \mathcal{F} \),

\[
G_M = e_M \mathcal{R}_V, \quad \text{where } G_M = \gcd(\det(\text{minor}_{\text{max}}(M))).
\]

**Proof.** By Corollary 4.1.1, there exists a matrix \( T \) such that

\[
\Theta = T \times M.
\]

Let \( k = \text{rank}(\Theta) \) and let \( C_1, \ldots, C_k \) be linearly independent columns of \( \Theta \). Consequently, the corresponding columns \( L_{i_1}, \ldots, L_{i_k} \) of \( M \), such that \( C_j = T \times L_{i_j} \), are also linearly independent.

---

\(^4\) The variety in question is not defined explicitly; one can use Kapur et al. (1994) or Buse et al. (2000) to define such varieties. In general, the associated variety is some projective closure of an affine set.

\(^5\) \( \mathbb{C}^* \) is \( \mathbb{C} - \{0\} \).
Let \( \nu \in V \) be a specialization of coefficients of the given polynomial system such that \( \text{rank}(\Theta(\nu)) \) is deficient; then \( G(\Theta(\nu)) = 0 \). Note that \( \mathcal{F}(x, \nu) \) has a solution in \( U \) if and only if \( \mathcal{R}_U(\nu) = 0 \).

Since \( \text{rank}(\Theta(\nu)) < k \), it follows that \( C_1(\nu), \ldots, C_k(\nu) \) are linearly dependent, which implies that

\[
\text{rank}(T(\nu)) < \text{rank}(T) \quad \text{or} \quad L_{ij}(\nu), \ldots, L_{ik}(\nu) \quad \text{are linearly dependent.}
\]

Since the above is true for any \( k \times k \) maximal minor of \( \Theta \), it follows that either (i) \( T(\nu) \) has deficient rank or (ii) every corresponding \( k \times k \) minor of \( M(\nu) \) has rank smaller than \( k \) implying that \( \text{rank}(M(\nu)) < \text{rank}(M) \). Therefore,

\[
\text{rank}(\Theta(\nu)) < \text{rank}(\Theta) \quad \implies \quad \text{rank}(T(\nu)) < \text{rank}(T) \quad \text{or} \quad \text{rank}(M(\nu)) < \text{rank}(M).
\]

If \( G_T = \gcd(\det(\text{minor}_{\text{max}}(T))) \) then

\[
\mathcal{R}_U | G_{\Theta} \quad \text{and} \quad G_{\Theta} | G_T G_M \quad \implies \quad \mathcal{R}_U | G_T G_M.
\]

To see that \( \mathcal{R}_U \) does not divide \( G_T \), note that generically, for \( n \)-degree unmixed polynomial systems (Kapur and Saxena, 1996; Saxena, 1997),

\[
\deg_{\mathcal{R}_U} G_T < \deg_{\mathcal{R}_U} G_{\Theta} = \deg_{\mathcal{R}_U} \mathcal{R}_U,
\]

where \( \deg_{\mathcal{R}_U} \) stands for the degree (in terms of generic coefficients of polynomial \( f_i \)) of the original polynomial system. Therefore, it must be the case that \( \mathcal{R}_U \) divides \( G_M \). \( \square \)

Another advantage of the Dixon dialytic matrix is that the choice of a polynomial \( g \) used to construct it influences not only the variety in question, and hence the particular resultant being computed, but also the size of the matrix itself. In practical cases, one might choose \( g \) so as to generate the smallest possible matrix and still have the condition for the existence of a solution in the variety under consideration.

This suggests that the projection operator extracted from the corresponding Dixon matrix typically contains more extraneous “information”. The variety over which the resultant is included in such a projection operator can be bigger. Further, the gcd of all maximal minors of \( T \) appears as a factor in the projection operator of the Dixon matrix.

### 4.4.1. Example

The following example illustrates the dependence of the resultant on the choice of a variety. Consider a mixed bivariate system,

\[
\begin{align*}
f_0 &= a_{10}x + a_{20}x^2 + a_{12}xy^2, \\
f_1 &= b_{01}y + b_{02}y^2 + b_{21}x^2, \\
f_2 &= u_1x + u_2y + u_0.
\end{align*}
\]

The twofold mixed volume of the above polynomial system is \( \langle 2, 2, 4 \rangle = 8 \) (see Section 2.1). Hence, generically the degree of the toric resultant (capturing toric solutions
When \( x = 0 \), \( f_0 \) becomes identically 0, and
\[
\begin{align*}
    f_1 &= b_0 y + b_2 y^2, \\
    f_2 &= u_2 y + u_0.
\end{align*}
\]
The resultant in that case is \( R_{x=0} = u_0 b_{02} - u_2 b_{01} \). Similarly, when \( y = 0 \), \( f_1 \) becomes identically 0, and
\[
\begin{align*}
    f_0 &= a_{10} x + a_{20} x^2, \\
    f_2 &= u_1 x + u_0.
\end{align*}
\]
The resultant in this case is \( R_{y=0} = u_0 a_{20} - u_1 a_{10} \). Therefore, the affine bounds are \( \langle 3, 3, 7 \rangle = 13 \). The degree of the resultant corresponding to zeroes in which either \( x = 0 \) or \( y = 0 \), written as \( R_{x=0 \ \text{or} \ y=0} \), is the mixed volume of \( (1, 1, 3) \). This is added to the degree of the toric resultant, which is
\[
R_T = u_2^4 b_{01} a_2^2 b_{21} + u_2^4 b_{21} a_2^2 b_{10} - u_2^3 b_{21} b_{02} a_2^2 u_0 + u_2^3 b_{21} b_{02} a_2^2 u_{10} \\
- 2u_2^2 b_{01} b_{21} u_{12} a_{10} + 4u_2^2 b_{01} b_{21} u_{12} a_{10} + 2u_2^2 b_{21} u_{02} a_{12} a_{10} \\
+ u_2 b_{02} u_{12} a_{20} b_{01} + 4u_2 b_{21} u_{12} u_{01} b_{02} a_{12} - 3u_2 b_{21} u_{12} b_{02} a_{12} u_0^2 \\
+ u_2^4 a_2^2 b_{01} + 2b_{21} u_{12} a_{12} b_{10} + b_{02} u_{12} a_{12} a_{10} + b_{02} u_{12} a_{12} u_0^2 - b_{02} u_{12} a_{12} u_{20} u_0.
\]
The resultant over the affine space is
\[
R_A = u_0 R_{x=0} R_{y=0} R_T.
\]
In contrast, the projection operator obtained from the \( 7 \times 7 \) Dixon matrix is
\[
\det(\Theta) = b_{02} a_{20} b_{21}^3 a_{12}^3 u_0 R_{x=0} R_{y=0} R_T.
\]
If a Dixon dialytic matrix is constructed using the term \( g = 1 \) in the construction, then
\[
\det(M) = b_{21} u_0 R_{x=0} R_{y=0} R_T.
\]
But if \( g = y \), for instance, is used to construct the Dixon dialytic matrix, then
\[
\det(M) = b_{21} R_{x=0} R_T.
\]
It appears that in a generic mixed case, the construction parameter \( g \) can be used to select an underlying variety for the resultant, if a priori knowledge about a solution space exists.

For toric solutions in \( (\mathbb{C}^*)^d \), it suffices to choose any monomial \( g \) for constructing the Dixon dialytic matrix. Selection of the monomial \( g \) can be formulated as an optimization problem as discussed below so that the Dixon dialytic matrix is of the smallest size. In this way, the degree of a projection operator and, hence, the degree of the extraneous factor in it can be minimized.

---

6 Here, by affine space, we mean \( \mathbb{C}^d \), and the resultant over it is a necessary and sufficient condition for the polynomial system to have a solution in \( \mathbb{C}^d \). In general \( \mathbb{C}^d \) is embedded in the smallest projective variety which contains an isomorphic image of \( \mathbb{C}^d \), and the resultant is defined over that projective variety.
5. Minimizing the degree of extraneous factors

For computing a resultant over a toric variety, the supports of a given polynomial system can be translated so as to construct smaller Dixon as well as Dixon dialytic matrices. This is evident from the following example for the bivariate case.

Example. Consider the following polynomial system:

\[ f_0 = a_{00} + a_{10}x + a_{01}y, \]
\[ f_1 = b_{02}y^2 + b_{20}x^2 + b_{31}x^3 y, \]
\[ f_2 = c_{00} + c_{12}x^2y + c_{21}x^3y. \]

This generic polynomial system has the twofold mixed volume of \( \langle 8, 3, 4 \rangle = 15 \); hence, the optimal dialytic matrix is \( 15 \times 15 \), containing eight rows from polynomial \( f_0 \), three rows from \( f_1 \) and four rows from \( f_2 \). Fig. 1 shows the overlaid supports of these polynomials.

To construct the Dixon dialytic matrix, if we choose \( g = x^\alpha y^\beta = 1, \) i.e., \( \alpha = (0, 0) \), and consider the original polynomial system \( \{f_0, f_1, f_2\} \), then

\[ |X_0| = 9, \quad |X_1| = 4, \quad |X_2| = 5, \]

and hence the Dixon dialytic matrix has 18 rows. In fact, for the polynomial system \( \{f_0, f_1, f_2\} \), the best choice for a term is based on its exponent vector being from \( \{(0, 0), (0, 1), (1, 0)\} \), each one producing a \( 18 \times 18 \) Dixon dialytic matrix. In other words, an extraneous factor of at least degree 3 is generated using the Dixon dialytic matrix no matter what multiplier monomial is used if supports are not translated.

On the other hand, if we consider \( \{x^2y^3f_0, f_1, xf_2\} \) and \( g \) is so chosen that the corresponding exponent vector is from \( \{(2, 1), (2, 2), (3, 1)\} \),

\[ |X_0| = 8, \quad |X_1| = 3, \quad |X_2| = 4, \]

and the resulting Dixon dialytic matrix has 15 rows, i.e., the matrix is optimal. Fig. 2 shows the translated supports.

This example also illustrates that it is possible to get exact resultant matrices if supports are translated even when untranslated supports have a non-empty intersection.

The Dixon matrix for the above polynomial system is of size \( 9 \times 9 \); its size is the same as though the system was unmixed with the support of the polynomial system being the union
of individual supports of the polynomials. For the translated polynomial system, however, the Dixon matrix is of size $8 \times 8$. In the two cases, there are extraneous factors of degree 12 and 9, respectively.

In fact, it will be shown (Section 5.1.1) that for generic mixed polynomial systems, the size of the Dixon matrix is at least $\max_{i=0}^{d}(|X_i|)$ when a monomial is appropriately chosen to do its construction.

As illustrated by the above example, the Dixon dialytic matrix and the Dixon matrix are sensitive to the translation of the supports of the polynomials in the polynomial system. Since the mixed volume of supports is invariant under translation, most resultant methods in which matrices are constructed using supports are also invariant with respect to translation of supports. Methods based on mixed volume compute resultants over toric variety and hence are insensitive to such translations of the supports of polynomials.

Since the Dixon dialytic matrix is sensitive to the choice of $g = x^\alpha$ (whereas the Dixon matrix is not), it is possible to further optimize the size of the Dixon dialytic matrix by properly selecting the construction parameter $g$ along with an appropriate translation of the supports of the polynomial system.

To formalize the above discussion, let $\alpha \in \mathbb{N}^d$; consider a translation $t = (t_0, t_1, \ldots, t_d)$ where $t_i \in \mathbb{N}^d$, to translate the support of a given polynomial system. The resulting translated support is denoted as $A + t$, standing for $(A_0 + t_0, A_1 + t_1, \ldots, A_d + t_d)$, where $A_i + t_i$ is the Minkowski sum.

Choosing an appropriate monomial $g = x^\alpha$ and $t$ for mixed polynomial systems can be formulated as an optimization problem in which the size of the support of each $\theta_i(g)$ and, hence, the size of the multiplier set for each $f_i$ is minimized.

One can either optimize the size of the Dixon matrix or, alternatively, the size of the Dixon dialytic matrix can be optimized. In other words, we can consider the following two problems:

(i) for the Dixon matrix construction, the size of the Dixon polynomial $|\Delta_{A+t_1}|$ is minimized;

(ii) for the Dixon dialytic matrix, where

$$\Phi_i(\alpha, t) = |\Delta_{(A_0+t_0, \ldots, A_{i-1}+t_{i-1}, \alpha, A_{i+1}+t_{i+1}, \ldots, A_d+t_d)}|.$$
the sum
\[ \Phi(\alpha, t) = \sum_{i=0}^{d} \Phi_i(\alpha, t), \]
i.e., the number of rows, is minimized.

In the case of (i), a heuristic for choosing such translation vectors \( t \) is described in Chcterba and Kapur (in press). For minimizing the size of Dixon dialytic matrices, a slightly modified method is needed, as the objective is to minimize the sizes of \( \theta_i(g) \), i.e., \( \Phi_i(\alpha, t) \) for each \( i \) (which also involves choosing \( g = x^\alpha \)). Since \( \Phi_i(\alpha, t) \) represents the number of rows corresponding to the polynomial \( x^\alpha f_i \) in the Dixon dialytic matrix, the goal is to find \( \alpha \) and \( t = \langle t_0, t_1, \ldots, t_d \rangle \) such that \( \Phi(\alpha, t) \) is minimized, that is, the size of the entire Dixon dialytic matrix is minimized so as to minimize the degree of the extraneous factor.

Below, we make some observations and prove properties which are helpful in selecting \( \alpha \) and \( t \).

5.1. Multiplier sets using the Dixon method

The multipliers used in the construction of a Dixon dialytic matrix are related to the monomials of the Dixon polynomial, which also label the columns of the Dixon matrix. For a given polynomial \( f_i \), its multiplier set generated using a term \( \alpha \) is obtained from \( \theta_i(f_0, \ldots, f_{i-1}, x^\alpha, f_{i+1}, \ldots, f_d) \). The support of the polynomial system for which \( \theta_i(f_0, \ldots, f_{i-1}, x^\alpha, f_{i+1}, \ldots, f_d) \) is the Dixon polynomial is \( \langle A_0, \ldots, A_{i-1}, \alpha, A_{i+1}, \ldots, A_d \rangle \). This support is denoted by \( A(i, \alpha) \).

Proposition 5.1. Given a support \( A = \langle A_0, A_1, \ldots, A_d \rangle \) of a polynomial system \( F \), for any \( \alpha \in \mathbb{N}^d \),
\[ \Delta_A \subseteq \bigcup_{i=0}^{d} \Delta_{A(i, \alpha)}, \]
i.e., the monomials of the associated Dixon matrix are a subset of the multipliers of the corresponding Dixon dialytic matrix.

Proof. Note that \( \Delta_{A(i, \alpha)} \) is the support of \( \theta_i(x^\alpha) \). Since \( x^\alpha \theta(f_0, \ldots, f_d) = \sum_{i=0}^{d} f_i \theta_i(x^\alpha) \), we can conclude that
\[ \Delta_A \subseteq \bigcup_{i=0}^{d} \Delta_{A(i, \alpha)}, \]
since no \( f_i \) has terms in variables \( x_j \).

As stated earlier, the Dixon polynomial depends on the variable order used in its construction. Let \( \Delta_A^{(x_1, x_2, \ldots, x_d)} \) stand for the support of the Dixon polynomial constructed using the variable order \( (x_1, x_2, \ldots, x_d) \), i.e., \( x_1 \) is first replaced by \( x_1 \), followed by \( x_2 \) and so on. Therefore,
\[ \Delta_A^{(x_d, \ldots, x_1)} \subseteq \bigcup_{i=0}^{d} \Delta_{A(i, \alpha)}^{(x_d, \ldots, x_1)}. \]
However, \( \Delta_A^{(x_0, \ldots, x_d)} = \Delta_A^{(x_1, \ldots, x_d)} \), as can be seen from Definition 3.1 of \( \theta \). Substituting this into the previous equation, the statement of the proposition is proved. □

The above proposition establishes that the support of the Dixon polynomial is contained in the union of the Dixon multiplier sets. It is shown below that the converse holds if the term \( g = x^\alpha \) chosen for the construction of the Dixon multipliers appears in all the polynomials of a given polynomial set.

**Theorem 5.1.** Given \( A = \langle A_0, A_1, \ldots, A_d \rangle \) as defined above, consider an \( \alpha \in \bigcap_{i=0}^d A_i \). Then, generically,

\[
\Delta_A = \bigcup_{i=0}^d \Delta_{A(i, \alpha)}.
\]

**Proof.** Using Proposition 5.1, it suffices to show that

\[
\bigcup_{i=0}^d \Delta_{A(i, \alpha)} \subseteq \Delta_A.
\]

Note that for generic polynomial systems

\[
\Delta_A = \bigcup_{\sigma \in A} \Delta_\sigma \quad \text{and} \quad \Delta_{A(i, \alpha)} = \bigcup_{\sigma \in A(i, \alpha)} \Delta_\sigma,
\]

and for every \( \sigma \in A(i, \alpha) \), \( \sigma \in A \). Hence the statement of the theorem follows. □

In particular, for an unmixed \( A \), where \( A_i = A_j \) for all \( i, j \in \{0,1,\ldots,d\} \),

\[
\Delta_A = \Delta_{A(i, \alpha)} \quad \text{for any } \alpha \in A_i, \quad \text{and} \quad i \in \{0,1,\ldots,d\}.
\]

The above property shows that the columns of the Dixon matrix are exactly the monomial multipliers of the Dixon dialytic matrix. That is,

\[
X \subseteq \bigcup_{i=0}^d X_i
\]

in the construction of a Dixon dialytic matrix. The above relation becomes an equality whenever the chosen monomial \( x^\alpha \) is in the support of all polynomials of an unmixed polynomial system. The above property is independent of the choice of the \( g = x^\alpha \) in \( A \), indicating a tight relationship between the Dixon matrix and the associated Dixon dialytic matrix.

**5.1.1. Size of the Dixon dialytic matrix**

Using Theorem 5.1, we can prove an observation made earlier that generically the size of a Dixon matrix is at least as big as the size of the largest multiplier set of the corresponding Dixon dialytic matrices.

**Theorem 5.2.** For a generic polynomial system \( F \), the size of the Dixon matrix is at least as big as the size of the largest multiplier set for the Dixon dialytic matrices, i.e.,

\[
\text{Size}(\theta) \geq \max_{i=0}^d \Phi_i(\alpha) \quad \text{when } \alpha \in \bigcap_{i=0}^d A_i.
\]
A direct consequence of the above theorem is:

**Corollary 5.2.1.** Consider a generic polynomial system with support $A = \{A_0, A_1, \ldots, A_d\}$ and let $\alpha \in \cap_{i=0}^d A_i$. Then,

$$(d + 1)\text{Size}(\Theta) \geq \text{Size}(M_\alpha),$$

where $M_\alpha$ is the Dixon dialytic matrix constructed using $g = x^\alpha$, and the above relation becomes an equality in the unmixed case, that is when $A_i = A_j$ for all $1 \leq i \neq j \leq d$.

**Proof.** The number of multipliers for a polynomial $f_i$ used in the construction of a Dixon dialytic matrix is $\Phi_i(\alpha)$, and in the unmixed case $\Phi_i(\alpha) = \Phi_j(\alpha)$, for all $i, j$. The number of rows of $M_\alpha$ is the sum of sizes of the multiplier sets for each polynomial, i.e. $\Phi(\alpha) = (d + 1)\Phi_i(\alpha)$ and also

$$\Delta_A = \bigcup_{i=0}^d \Delta_{A(i,\alpha)}$$

and therefore $|\Delta_{A(i,\alpha)}| = \Phi_i(\alpha) \leq |\Delta_A| = \text{Size}(\Theta)$.

By Theorem 5.1. It follows that $M_\alpha$ is at most $d + 1$ times bigger than $\Theta$, where there is equality in the unmixed case. \qed

Using the above propositions, we have one of the key results of this paper.

**Theorem 5.3.** Given a generic, unmixed polynomial system $F$ and a monomial $x^\alpha$ in $F$, if the Dixon matrix is exact, the Dixon dialytic matrix built using monomial $x^\alpha$ is also exact.

**Proof.** Since the Dixon matrix is exact, its size $\text{Size}(\Theta)$ equals the degree of the coefficients of $f_i$ appearing in the resultant, which, by Theorem 5.1, is the size of multiplier set for each polynomial $f_i$ in the polynomial system $F$. Hence the coefficients of $f_i$ will appear at most of the same degree $\text{Size}(\Theta)$, in the projection operator of the Dixon dialytic matrix. But this is precisely their degree in the resultant; therefore the projection operator extracted from the Dixon dialytic matrix is precisely the resultant, that is, $M_\alpha$ is exact. \qed

In all generic cases, the ratio between the sizes of the two matrices is at most $d + 1$; therefore, the Dixon dialytic matrices are as good as the Dixon matrices in unmixed cases as regards the absence of extraneous factors; usually, the Dixon dialytic matrices are better in mixed cases.

### 5.2. Choosing the polynomial parameter for constructing multiplier sets

From the last two subsections, we can make the following observations on designing a heuristic to choose $g = x^\alpha$ and $t$ to minimize $\Phi(\alpha, t)$.

Notice that if $g$ is generic then it is sufficient to select $g$ as a monomial, since

$$g^\Theta(f_0, f_1, \ldots, f_d) = \sum_{a \in G} g_a x^\alpha \theta(f_0, f_1, \ldots, f_d)$$

$$= \sum_{i=0}^d \theta_i \left( f_0, \ldots, f_{i-1}, \sum_{a \in G} g_a x^\alpha, f_{i+1}, \ldots, f_d \right)$$

\[ g(\alpha) = \sum_{\alpha \in G} g_{\alpha} x^\alpha \] and, hence, the monomial multipliers in the construction of the Dixon dialytic matrix are

\[ X_i = \Delta_{A(i,G)} = \bigcup_{\alpha \in G} \Delta_{A(i,\{\alpha\})}. \]

Since our goal is to build smaller Dixon matrices, i.e. have smaller multiplier sets, in the generic case, it is not useful to consider parameter \( g \) as a general polynomial since a single monomial will suffice.

Choosing an appropriate monomial parameter \( g = x^\alpha \), i.e. \( G = \{\alpha\} \) and translation \( t \) for mixed polynomial systems can be formulated as an optimization problem for the Dixon dialytic matrix, in which the size of the support of each \( \theta_i(g) \) and, hence, the size of the multiplier set for each \( f_i \) is minimized.

1. Let \( Q_i = \bigcup_{j=0}^{d} (t_j + A_j) \); consider the support \( P_i = \langle Q_0, \ldots, Q_i \rangle \) of an unmixed polynomial system. Then, by Theorem 5.1,

\[ \bigcup_{i=0}^{d} \Delta_{[A+i|i,a]}(i,a) \subseteq \Delta_{P_i} \] and therefore for all \( i \), \( \Phi_i(a, t) \leq |\Delta_{P_i}|. \)

It is difficult to minimize the size of \( \Delta_{[A+i|i,a]}(i,a) \), which is \( \Phi_i(a, t) \), as it depends on the selection of \( t_j \) for \( j \neq i \), and optimal selection for \( \Phi_i(a, t) \) might be worst selection for \( \Phi_j(a, t) \) when \( i \neq j \). Instead we will minimize \( |\Delta_{P_i}| \), which is upper bound on all \( \Phi_i(a, t) \). To minimize \( |\Delta_{P_i}| \) we can use same method to choose \( t \) as in the case of minimizing the size of the Dixon matrix. Consider the following incremental procedure:

(a) Let \( Q^i_i = \bigcup_{j=0}^{d} (t_j + A_j) \), and let \( P^i_i = \langle Q^0_i, \ldots, Q^d_i \rangle \); find \( t_{i+1} \) such that the \( \Delta_{P_{i+1}} \) is minimal.

(b) The procedure for finding \( t_{i+1} \) is iterative and like a gradient ascent (hill climbing) method. Starting with \( i = 0 \), set the initial guess to be \( t_{i+1} = (0, \ldots, 0) \), and compute the size of \( \Delta_{P_{i+1}} \). Then, select a neighboring point of \( t_{i+1} \) and compute the size of the resulting set \( \Delta_{P_{i+1}} \). If the set is smaller, select the neighboring point. Stop when all neighboring points result in support hulls of bigger size.

2. Once \( t \) is fixed by using the above procedure to minimize \( |\Delta_{P_i}| \), search for a monomial with exponent \( \alpha \) for constructing the Dixon dialytic matrix such that

\[ \alpha \in \text{SupportHull} \bigcup_{j=0}^{d} (t_j + A_j), \]

for maximal number of indices \( i = 0, 1, \ldots, d \), where SupportHull is defined below.
The support hull of a given support is similar to the associated convex hull. (For a complete description, see Chtcherba, 2003.)

**Definition 5.1.** Given \( k \in \mathbb{Z}^d_2 \) and points \( p, q \in \mathbb{N}^d \) define

\[
p \overset{k}{\succeq} q \quad \text{if} \quad \begin{cases} p_j \leq q_j & \text{if } k_j = 1, \\ p_j \geq q_j & \text{if } k_j = 0. \end{cases}
\]

The support hull can be defined to be the set of points which are “inside” the hull.

**Definition 5.2.** Given a support \( \mathcal{P} \subset \mathbb{N}^d \), define its support hull to be

\[
\text{SupportHull}(\mathcal{P}) = \{ p \mid \forall k \in \mathbb{Z}^d_2, \exists q \in \mathcal{P}, \text{ such that } p \overset{k}{\succeq} q \}.
\]

The search is first done for the translation vector \( t \), as the choice of \( \alpha \) depends on the fixed value for \( t \). Below, we illustrate the above procedure for constructing smaller Dixon dialytic matrices.

**Example.** Consider a highly mixed polynomial system (see Fig. 3). The above procedure is illustrated in some detail on this example.

\[
\begin{align*}
f_0 &= a_{20}x^2 + a_{40}x^4 + a_{56}x^5y^6 + a_{96}x^9y^6, \\
f_1 &= b_{30}x^3 + b_{27}x^2y^7 + b_{59}x^5y^9 + b_{69}x^6y^9, \\
f_2 &= c_{04}y^4 + c_{29}x^2y^9 + c_{88}x^8y^8,
\end{align*}
\]

where \( x, y \) are variables and \( a_{ij}, b_{ij} \) and \( c_{ij} \) are variables.

The objective is to translate the supports so that they overlap the most. \( \mathcal{A}_0 \) is fixed; we try to adjust \( \mathcal{A}_1 \). At the starting point \((t_0 = (0, 0))\), \( \Delta_{\mathcal{P}_1} \) is obtained, which results in an unmixed system whose Dixon matrix is of size 80 columns. Using that as the starting point, we do a local search in all four directions, and pick the one with the least matrix size: 74.

We repeat the procedure as illustrated below (see Fig. 4), finally leading to the case where \( t_0 = (-1, 3) \) and the Dixon matrix size is 65. At this point, all four directions lead to matrices of larger size, which is the stopping criterion.

Now that \( t_0 \) and \( t_1 \) are fixed (in the above case they are \((-1, 3)\) and \((0, 0))\), \( t_2 \) is found in the same way. Starting with the initial value for which the Dixon matrix is of size 97, we
Fig. 4. Minimizing the upper bound on the Dixon dialytic matrix size by adjusting $t_1$ and then $t_2$.

Fig. 5. Adjusted $A_0$.

eventually get the Dixon matrix of size 82 at $t_2 = (2, 0)$ as shown above (Fig. 5). Therefore we get $t_2 = (2, 0)$. Fig. 6 shows the final arrangement of the supports.

For a proper choice of $\alpha$ (see Fig. 7, where $\Phi(\alpha, t) = \sum_{i=0}^{d} \Phi_i(\alpha, t)$ is shown), $\Phi_1(\alpha, t) \leq 82$. We should note that the optimal choice (obtained by exhaustive searching) for the Dixon dialytic matrix is $t = ((-1, 3), (0, 0), (3, 0))$. In the above example, originally the Dixon matrix (without translating supports) has 109 columns; after translation, it has only 82 columns. The Dixon dialytic matrix for the original supports is $91 + 95 + 89 = 275$ rows; after the above translation, the number of rows is $77 + 53 + 65 = 195$. The optimal translation leads to a Dixon dialytic matrix with $75 + 53 + 65 = 193$ rows.

Note that BKK bound of the above system is $(75, 51, 63)$, putting the lower bound of 189 on the size of any dialytic resultant matrix. By trying to optimize the size of the matrix,
the degree (and as a consequence the size) of the extraneous factors has been brought down. In this example, it is \((2, 2, 2)\), where the \(i\)th entry in the tuple denotes the degree of extraneous factor in terms of coefficients of \(f_i\). Note that in this example, \(\alpha\) is chosen from the support hull intersection; in general, good choices for \(\alpha\) are always from the support hull, as the Dixon matrix construction is invariant in the presence of monomials whose exponent is support hull interior (see Chcterba and Kapur, in press).

In general, to find a translation vector \(t = \langle t_0, \ldots, t_d \rangle\) using the above procedure, one possibility is to search for each \(t_i\) in the range so that there is some overlap between \(Q^{i-1}_i\) and \(t_i + A_i\). If the maximum degree of the polynomial system is \(k\), then the distance from optimal \(t_i\) and the initial guess will be in the order \(k\). Hence the cost of finding translation vector \(t\) is

\[ O(dk)\text{Cost}(\Delta P_i). \]
In the next section we will derive the complexity of constructing a Dixon matrix and a Dixon dialytic matrix as well as $\text{Cost}(|\Delta P_j|)$.

6. Complexity and empirical results

Proposition 6.1. Given a polynomial system $F = \{f_0, f_1, \ldots, f_d\}$ with support $\langle A_0, A_1, \ldots, A_d \rangle$, let $n = |\bigcup_{i=0}^{d} A_i|$: the complexity of constructing the Dixon dialytic matrix $M_\alpha$ for fixed $\alpha \in \mathbb{N}^d$ is

$$T_{DM} = O \left( \frac{n^d d^2 (d+1)}{(n-d)!} \right) = O(d^3 n^d).$$

Proof. Assume (in the worst case) that each support has $n$ points; therefore, each $\theta_i$ has the same structure, except for different permutation of columns. Hence, the total complexity is bounded from above by $(d+1)$ times the complexity of expanding single $\theta_i$ (w.l.o.g. assume $\theta_0$):

$$\theta_0(x^\alpha) = \theta(x^\alpha, f_1, \ldots, f_d)$$

$$= \prod_{i=0}^{d} \frac{1}{x_i - x_i} \sum_{\sigma \in (\langle A \rangle, A_1, \ldots, A_d)} \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 \\
c_1,\sigma_0 & c_1,\sigma_1 & c_1,\sigma_2 & \cdots & c_1,\sigma_d \\
c_2,\sigma_0 & c_2,\sigma_1 & c_2,\sigma_2 & \cdots & c_2,\sigma_d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_d,\sigma_0 & c_d,\sigma_1 & c_d,\sigma_2 & \cdots & c_d,\sigma_d \\
\end{vmatrix}$$

$$\times \begin{vmatrix}
\pi_0(x^\alpha) & \pi_0(x)\sigma_1 & \pi_0(x)\sigma_2 & \cdots & \pi_0(x)\sigma_d \\
\pi_1(x^\alpha) & \pi_1(x)\sigma_1 & \pi_1(x)\sigma_2 & \cdots & \pi_1(x)\sigma_d \\
\pi_2(x^\alpha) & \pi_2(x)\sigma_1 & \pi_2(x)\sigma_2 & \cdots & \pi_2(x)\sigma_d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\pi_d(x^\alpha) & \pi_d(x)\sigma_1 & \pi_d(x)\sigma_2 & \cdots & \pi_d(x)\sigma_d \\
\end{vmatrix},$$

where $\sigma_i \in A_i$ for $i \in \{1, \ldots, d\}$ and $\sigma_i \neq \sigma_j$ for $i \neq j$. Hence there are $\binom{d}{2}$ terms in this sum, where we need to expand two determinants of size $(d+1)$. In the worst case, it will take $2(d+1)!$ time. It is not necessary to carry out division by $x_i - x_i$, as the resulting monomials can be easily deduced. This is analogous to the Sylvester matrix construction, where given the degrees of the polynomials, any entry in the matrix can be deduced in constant time; in the second determinant, operations have to be done on exponent vectors and, hence, have the complexity of $d$. \qed

If we take into account the search for optimizing translation vector $t$, and constructing the monomial with exponent $\alpha$, then the total complexity is

$$O(d^4 r n^d),$$

where $r$ is the range of search for the translation vector $t$, which is bounded by the difference between the minimum and maximum degrees of all polynomials.

In Canny and Emiris (2000) and Emiris and Canny (1995), Canny and Emiris proposed two methods, subdivision and incremental, for constructing sparse resultant matrices from
Table 1
Performance comparison of Dixon dialytic construction versus subdivision and incremental constructions of Newton sparse matrices

<table>
<thead>
<tr>
<th>No</th>
<th>Problem</th>
<th>Incremental</th>
<th>Subdivision</th>
<th>$M_0$</th>
<th>$M_{(a,t)}$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Size</td>
<td>Time</td>
<td>Size</td>
<td>Time</td>
</tr>
<tr>
<td>1</td>
<td>Pappus’s theorem ($6d$)</td>
<td>–</td>
<td>–</td>
<td>20</td>
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<td>2</td>
<td>Max. volume of tetrahedron ($4d$)</td>
<td>107</td>
<td>5.75</td>
<td>137</td>
<td>186.83</td>
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<td>3</td>
<td>Side bisector ($2d$)</td>
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<td>0.08</td>
<td>33</td>
<td>7.58</td>
</tr>
<tr>
<td>4</td>
<td>Conformal anal., cyclic molecules ($3d$)</td>
<td>112</td>
<td>4.14</td>
<td>119</td>
<td>140.96</td>
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<tr>
<td>5</td>
<td>Kissing circles theorem ($5d$)</td>
<td>–</td>
<td>–</td>
<td>239</td>
<td>624.62</td>
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<tr>
<td>6</td>
<td>Implicitization of strophoid ($2d$)</td>
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<td>0.12</td>
<td>11</td>
<td>3.56</td>
</tr>
<tr>
<td>7</td>
<td>Random unmixed ($2d$), 21 unmixed</td>
<td>395</td>
<td>63.77</td>
<td>384</td>
<td>881.26</td>
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<tr>
<td>8</td>
<td>Random mixed ($3d$), 4 simplexes</td>
<td>461</td>
<td>46.50</td>
<td>480</td>
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<tr>
<td>9</td>
<td>Random mixed ($2d$), mixed</td>
<td>301</td>
<td>19.06</td>
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<td>Random mixed ($4d$), $\leq 3$ simplexes</td>
<td>745</td>
<td>188.52</td>
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<td>4738.70</td>
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Table 2
Interpolation timings with nine parameters

<table>
<thead>
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<th>Dialytic matrix size</th>
<th>Interpolation time (s)</th>
<th>Rate (evaluations/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>1.37</td>
<td>3300</td>
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<tr>
<td>22</td>
<td>12.25</td>
<td>1100</td>
</tr>
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<td>29</td>
<td>52.30</td>
<td>600</td>
</tr>
<tr>
<td>36</td>
<td>173.40</td>
<td>290</td>
</tr>
<tr>
<td>44</td>
<td>1099.80</td>
<td>150</td>
</tr>
<tr>
<td>52</td>
<td>3530.91</td>
<td>87</td>
</tr>
</tbody>
</table>

the support of a polynomial system. The subdivision and incremental methods have, respectively, complexities as follows:

$$T_S = O\left(\frac{\text{Size}(M)d^{0.5}n^{6.5} \log^2 k \log^2 \frac{1}{\epsilon_l \epsilon_\delta}}{\epsilon_l \epsilon_\delta}\right)$$

and

$$T_I = O^*(e^{3d}n^{5.5}(\deg R)^3) + O^*(d^{7.5}n^{2d+5.5})$$,
where \( k \) is the maximum degree of the polynomials in the system, \( \epsilon_i \) is the probability of failure to pick the generic lifting vector and \( \epsilon_\delta \) is the probability of perturbation failure (Canny and Emiris, 2000). In the above formula, \( \text{Size}(M) = \Omega(\deg R) \). With this crude lower bound for the size of the resultant matrix and the degree of the resultant itself, we can compare the methods:

\[
\frac{T_S}{T_{DD}} = O\left(\frac{\text{Size}(M)d^{6.5}n^{6.5} \log^2 k \log^2 \frac{1}{\epsilon_\delta}}{n^d}\right)
\]

and

\[
\frac{T_I}{T_{DD}} = O^*\left(\frac{e^{3d}n^{5.5}(\deg R)^3}{d^3n^d}\right) + O^*(d^{4.5}n^{d+5.5}),
\]

where \( T_{DD} \) is the construction complexity of the Dixon Dialytic matrix.

The experimental results\(^7\) below confirm that the Dixon based dialytic method is an order of magnitude faster than the subdivision and incremental algorithms as well as being more successful in constructing smaller matrices (the main bottleneck); furthermore, it can be optimized to construct even smaller matrices if one is willing to spend more time in searching for the appropriate \( \alpha \) and \( t \).

Column \( M_0 \) shows the size and construction time when \( t = \langle 0, \ldots, 0 \rangle \) and \( \alpha = 0 \). Column \( M_{(\alpha,t)} \) shows results when \( t \) and \( \alpha \) are searched for using the above heuristic. The entry containing \( \text{Unmix} \) is not filled in since the example is unmixed (for which no optimization is needed).

The implementation of the \textit{incremental method} (Emiris and Canny, 1995) is in C, whereas other algorithms are implemented in Maple. The implementation of the \textit{subdivision} algorithm used is downloaded from I. Emiris’s web site.\(^8\) The optimization algorithm is implemented very crudely, without taking advantage of appropriate data structures. All operations are done on lists; hence, timings deteriorate fast with the number of variables. The implementation for constructing the Dixon as well as Dixon dialytic matrices can be obtained from http://www.cs.panam.edu/\textasciitilde cherba.

The advantages of optimizing vectors \( t \) and \( \alpha \) are greater as the input system is more and more mixed. Since the method for finding \( t \) and \( \alpha \) is based on a heuristic, it is not guaranteed to produce optimal values. In Example 6 (see Table 1), for instance, the original values from the Dixon dialytic algorithm already give an optimal matrix; the heuristic on the other hand gives a worse value. This is mainly due to the fact that optimization is done assuming generic coefficients.

The above heuristic appears to be expensive in time; for instance, for Example 10, the size of the matrix reduces from 579 to 419 after spending three orders of magnitude more time in finding the appropriate translation vector and the monomial for construction. Table 2 shows the correlation between the time taken to interpolate a determinant and the

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\(^7\) These experiments were carried out on a machine with a 700 MHz AMD Athlon processor with 384 MB of RAM.

\(^8\) http://cgi.di.uoa.gr/\textasciitilde emiris/index-eng.html.
size of a matrix. As can be seen, the size of the resultant matrix produces a major bottleneck in the determinant computation.

7. Multigraded polynomial systems: exact cases

In this section, we prove that the above construction of Cayley–Dixon dialytic matrices is exact for a multigraded system, i.e., the determinant of such a matrix is indeed the resultant. See Sturmfels and Zelevinski (1994) and Dickenstein and Emiris (2003) for theoretical treatment as well as see Chtcherba and Kapur (2000a) for analysis of the Cayley–Dixon construction for multigraded systems.

Below we consider only unmixed polynomial systems with supports $A = \langle A_0, A_1, \ldots, A_d \rangle$, where $A_i = A_j$. For clarity, we will drop the index and denote by $A$ the support of each polynomial, i.e., $A = A_i$ for all $i \in \{0, 1, \ldots, d\}$.

One of the operations on supports which was considered in Chtcherba and Kapur (2000a) is the direct sum on supports.

**Definition 7.1.** Given two supports $P \subset \mathbb{N}^k$ and $Q \subset \mathbb{N}^l$, define the **direct sum** of $P$ and $Q$:

$$P \oplus Q = \{(p_1, \ldots, p_i, q_1, \ldots, q_k) \mid p = (p_1, \ldots, p_i) \in P \text{ and } q = (q_1, \ldots, q_k) \in Q\}.$$

A polynomial is called **homogeneous** if all monomials appearing in the polynomial have the same degree. In terms of the support $A$, a polynomial is homogeneous of degree $n$ if for $\alpha = (\alpha_1, \ldots, \alpha_d) \in A$, $\alpha_1 + \cdots + \alpha_d = n$. Any polynomial can be homogenized by introducing an extra variable. In a sense, each polynomial has a homogeneous and non-homogeneous version.

**Definition 7.2.** A polynomial with support $A$ is called **multihomogeneous** of type $(l_1, l_2, \ldots, l_r; k_1, k_2, \ldots, k_r)$ for some integers $l_i$, $k_j$ and $r$ if $A \subseteq Q_1 \oplus \cdots \oplus Q_r$, and $Q_i \subset \mathbb{N}^{l_i}$ is the support of a polynomial of degree $k_i$, for $i = 1, \ldots, r$.

By abuse of notation, we call a polynomial multihomogeneous of a certain type if it can be homogenized into one. Note that the same polynomial can be multihomogenized in a number of different ways. For example, the polynomial

$$f = c_1x + c_2y + c_3$$

can be homogenized into

$$f = c_1xt + c_2ys + c_3st$$

where the first is of type $(2; 1)$ in terms of variables $x, y$ with the homogenizing variable $z$ and the second is of type $(1; 1, 1)$ in terms of variables $x, y$ and the homogenizing variables $s$ and $t$, respectively. Note that in the first case, all monomials of degree 1 are present, whereas in the second case, the monomial $xy$ could have been included with the resulting polynomial being still of type $(1, 1; 1, 1)$. 
Proposition 7.1. Given a unmixed generic polynomial system with polynomial support \( A \subseteq P \cup Q \), where \( P \subset \mathbb{N}^k \) and \( Q \subset \mathbb{N}^l \),

\[
\Delta_A \subseteq \Delta_P \oplus \left( \bigoplus_{k} Q + \cdots + Q + \Delta_Q \right).
\]

Proof. A polynomial with support \( A \) has two blocks of variables—one corresponding to \( P, \{x_1, \ldots, x_k\} \), and the other corresponding to \( Q, \{y_1, \ldots, y_l\} \). Obviously, the polynomial is in variables \( \{x_1, \ldots, x_k, y_1, \ldots, y_l\} \). The matrix for computing the Dixon polynomial can be split according to \( P \) and \( Q \):

\[
\theta(f_0, \ldots, f_d) = \left( \prod_{i=1}^{k} \frac{1}{x_j - x_i} \prod_{i=1}^{l} \frac{1}{y_j - y_i} \right) \times \begin{vmatrix}
\pi_0(f_0) & \pi_0(f_1) & \cdots & \pi_0(f_d) \\
\vdots & \vdots & & \vdots \\
\pi_{k-1}(f_0) & \pi_{k-1}(f_1) & \cdots & \pi_{k-1}(f_d) \\
\pi_k(f_0) & \pi_k(f_1) & \cdots & \pi_k(f_d) \\
\vdots & \vdots & & \vdots \\
\pi_{k+l-1}(f_0) & \pi_{k+l-1}(f_1) & \cdots & \pi_{k+l-1}(f_d) \\
\pi_{k+l}(f_0) & \pi_{k+l}(f_1) & \cdots & \pi_{k+l}(f_d)
\end{vmatrix}.
\]

Let \( J = \{0, \ldots, k+l\} \); for a subset \( a = \{a_1, \ldots, a_k\} \subset J \) of size \( k \), let \( \hat{a} \) be such that \( a \cup \hat{a} = J \), i.e., \( |\hat{a}| = l \). Using the Laplace formula for the determinant, the above expression can be expanded in terms of the determinants of minors coming from each block as

\[
\theta(f_0, \ldots, f_d) = \sum_{a \subset J} (-1)^{|a|} \left( \prod_{i=1}^{k} \frac{1}{x_j - x_i} \right) \times \begin{vmatrix}
\pi_0(f_{a_1}) & \pi_0(f_{a_2}) & \cdots & \pi_0(f_{a_k}) \\
\vdots & \vdots & & \vdots \\
\pi_{k-1}(f_{a_1}) & \pi_{k-1}(f_{a_2}) & \cdots & \pi_{k-1}(f_{a_k}) \\
\pi_k(f_{a_1}) & \pi_k(f_{a_2}) & \cdots & \pi_k(f_{a_k}) \\
\vdots & \vdots & & \vdots \\
\pi_{k+l-1}(f_{a_1}) & \pi_{k+l-1}(f_{a_2}) & \cdots & \pi_{k+l-1}(f_{a_k}) \\
\pi_{k+l}(f_{a_1}) & \pi_{k+l}(f_{a_2}) & \cdots & \pi_{k+l}(f_{a_k})
\end{vmatrix}.
\]

The support of the first determinant in the above expression in terms of variables \( \{x_1, \ldots, x_d\} \) is contained in \( \Delta_P \). Also it is not hard to see that the support of the first determinant in terms of variables \( \{y_1, \ldots, y_d\} \) is contained in \( \bigoplus_{k} Q + \cdots + Q \). Hence the support of the first determinant is contained in

\[
\Delta_P \oplus \left( \bigoplus_{k} Q + \cdots + Q \right).
\]
The support of the second determinant in terms of variables \(\{y_1, \ldots, y_l\}\) is exactly \(\Delta_Q\). (Note that the second determinant does not contain any variables from \(\{x_1, \ldots, x_d\}\).)

Combining these, we get that

\[
\Delta_A \subseteq \Delta_P \oplus \left( \underbrace{Q + \cdots + Q}_k + \Delta_Q \right).
\]

We consider a subclass of multihomogeneous polynomials, called **multigraded** polynomials.

**Definition 7.3.** A multihomogeneous polynomial of type \((l_1, l_2, \ldots, l_r; k_1, k_2, \ldots, k_r)\) is called **multigraded** if for each \(i = 1, \ldots, r\), either \(l_i = 1\) or \(k_i = 1\).

A **multigraded** polynomial system \(\mathcal{F}\) of type \((l_1, l_2, \ldots, l_r; k_1, k_2, \ldots, k_r)\) is an unmixed system of \(d+1\) generic multigraded polynomials in \(d\) variables, where \(\sum_{i=1}^r l_i = d\). We prove below that the Dixon dialytic matrix for a multigraded system is square, and its determinant is the resultant of \(\mathcal{F}\).

**Proposition 7.2.** Let \(\mathcal{F}\) be a multigraded polynomial system of type \((l; k)\); then

\[
\Delta_A = \{ p = (p_1, \ldots, p_l) \mid p_1 + \cdots + p_l \leq k-1 \}.
\]

**Proof.** If \(l = 1\), then the support of the Dixon polynomial corresponds to the monomials \(\{1, x, x^2, \ldots, x^{k-1}\}\) used in constructing the Bézout matrix. If \(k = 1\), then \(\mathcal{F}\) is a linear system of equations. The Dixon polynomial in that case contains a constant term, obtained by expanding the corresponding determinant and performing division by \(x_i - x_i\). □

**Proposition 7.3.** Let \(\mathcal{A}\) be the support of a multihomogeneous polynomial of type \((l; k)\). For an integer \(n > 0\), \(\mathcal{Q} = \underbrace{\mathcal{A} + \cdots + \mathcal{A}}_n\) is the support of a multihomogeneous polynomial of type \((l; nk)\), that is

\[
\mathcal{Q} = \{ p = (p_1, \ldots, p_l) \mid p_1 + \cdots + p_l \leq nk \} \quad \text{and} \quad |\mathcal{Q}| = \binom{nk+l}{l}.
\]

**Proof.** Since \(\mathcal{A} = \{ p = (p_1, \ldots, p_l) \mid p_1 + \cdots + p_l \leq k \}\), the sum of \(\mathcal{A}\), \(n\) times, contains all points up to \(nk\), which is the support of a multihomogeneous polynomial of type \((l; nk)\). The number of points in the support is \(|\mathcal{Q}| = \binom{nk+l}{l}\). □

From the last two propositions, we have:

**Proposition 7.4.** Let \(\mathcal{F}\) be a multigraded polynomial system of type \((l; k)\) with support \(\mathcal{A}\). Then

\[
\underbrace{\mathcal{A} + \cdots + \mathcal{A}}_n + \Delta_A = \{ p = (p_1, \ldots, p_l) \mid p_1 + \cdots + p_l \leq nk + k - 1 \}.
\]
Definition 7.4. Given a partition of variables \((l_1, \ldots, l_r)\), let
\[
(m_1, \ldots, m_r) = \left\{ \left( p_{1,1}, \ldots, p_{1,l_1}, p_{2,1}, \ldots, p_{2,l_2}, \ldots, p_{r,1}, \ldots, p_{r,l_r} \right) \middle| \sum_{j=1}^{l_i} p_{i,j} \leq m_i, i = 1, \ldots, r \right\}.
\]
The \(r\)-tuple \((m_1, \ldots, m_r)\) is called a **multi-index**.

It is easy to see that
\[
|\(m_1, \ldots, m_r)\| = \prod_{i=1}^{r} \binom{m_i + l_i}{l_i}.
\]

Proposition 7.5. Given a multigraded polynomial system \(F = \{f_0, f_1, \ldots, f_d\}\) of type \((l_1, l_2, \ldots, l_r; k_1, k_2, \ldots, k_r)\),
\[
\Delta_A \subseteq (m_1, m_2, \ldots, m_r), \quad \text{where } m_i = k_i - 1 + k_i \sum_{j=1}^{i-1} l_j.
\]

**Proof.** This directly follows from the previous two propositions. □

Let \(M_0\) be the Dixon dialytic matrix constructed from a multigraded polynomial system \(F\) using monomial \(x^0 = 1\). Since for the unmixed case \(\Delta_A(i,0) = \Delta_A\),
\[
\#\text{rows}(M_0) \leq \left( 1 + \sum_{i=1}^{r} l_i \right) |(m_1, m_2, \ldots, m_r)|
\]
\[
= \left( 1 + \sum_{i=1}^{r} l_i \right) \prod_{i=1}^{r} \binom{m_i + l_i}{l_i} = (d + 1) \prod_{i=1}^{r} \binom{m_i + l_i}{l_i}.
\]
given that \(d = \sum_{i=1}^{r} l_i\).

Proposition 7.6. Let \(X = (m_1, \ldots, m_r)\) and let \(f\) be a multihomogeneous polynomial of type \((l_1, l_2, \ldots, l_r; k_1, k_2, \ldots, k_r)\) with support \(\mathcal{P}\). Then, \(X + \mathcal{P} = (m_1 + k_1, \ldots, m_r + k_r)\).

Thus, the number of columns in \(M_0\) is at most
\[
|(m_1 + k_1, m_2 + k_2, \ldots, m_r + k_r)| = \prod_{i=1}^{r} \binom{m_i + k_i + l_i}{l_i}.
\]

The degree of the resultant \(R\) of \(F\) is determined by the mixed volumes of the system (see Sturmfels and Zelevinski, 1994), and is given by
\[
\deg R = (d + 1) \text{MxVol}(S_X(F)) = (d + 1) d! \text{Vol}(S_X(F)) = (d + 1) d! \prod_{i=1}^{r} \frac{k_i^{l_i}}{l_i!}.
\]
Lemma 7.1. For a multigraded system \( \mathcal{F} \), let \( m_i = k_i - 1 + k_i \sum_{j=1}^{i-1} l_j \).

(i) The determinant of a maximal minor of \( M_0 \) is a projection operator, which has total degree bigger than or equal to the resultant \( R \), i.e.,

\[
(d + 1) \prod_{i=1}^{r} \binom{m_i + l_i}{l_i} \geq \#_{\text{rows}}(M_0) \geq \deg R.
\]

and

\[
\prod_{i=1}^{r} \binom{m_i + k_i + l_i}{l_i} \geq \#_{\text{cols}}(M_0) \geq \deg R.
\]

(ii) The number of rows in \( M_0 \) is equal to the (total) degree of the resultant \( R \), i.e.,

\[
(d + 1) \prod_{i=1}^{r} \binom{m_i + l_i}{l_i} = (d + 1) d! \prod_{i=1}^{r} \frac{k_i}{l_i}.
\]

(iii) The number of columns in \( M_0 \) is equal to the number of rows in \( M_0 \); hence the matrix \( M_0 \) is square, i.e.,

\[
(d + 1) \prod_{i=1}^{r} \binom{m_i + l_i}{l_i} = \prod_{i=1}^{r} \binom{m_i + k_i + l_i}{l_i}.
\]

Proof. (i) This is a consequence of a property of the Dixon matrix (Kapur et al., 1994; Saxena, 1997) that the determinant of its maximal minor contains a toric resultant. By Theorem 4.3, the Dixon dialytic matrix \( M_0 \) has that property as well.

For (ii), we have from (i) that

\[
(d + 1) \prod_{i=1}^{r} \binom{m_i + l_i}{l_i} \geq \#_{\text{rows}}(M_0) \geq \deg R = (d + 1) d! \prod_{i=1}^{r} \frac{k_i}{l_i}.
\]

Below we show that for multigraded systems (i.e., for each \( i, k_i = 1 \) or \( l_i = 1 \)), the lower and upper bounds on the number of rows in \( M_0 \) coincide. Consider

\[
h_i = \binom{m_i + l_i}{l_i} \left| \frac{t_i k_i}{k_i^i / l_i^i} \right|_{k_i = 1 \text{ or } l_i = 1} = \frac{t_i!}{l_i - 1}.
\]

where \( t_i = \sum_{j=1}^{i} l_j \). Thus, \( \prod_{i=1}^{r} h_i = t_r! = d! \), proving (ii).

For (iii), for multigraded systems (i.e., for each \( i, k_i = 1 \) or \( l_i = 1 \)), consider

\[
g_i = \frac{(m_i + k_i + l_i)}{(m_i + l_i)} \left| \frac{(t_i + 1) k_i}{t_i k_i / l_i} \right|_{k_i = 1 \text{ or } l_i = 1} = \frac{t_i + 1}{l_i - 1 + 1}
\]

The product \( \prod_{i=1}^{r} g_i = t_r + 1 = l_1 + \cdots + l_r + 1 = d + 1 \), and hence, (iii) follows. \( \square \)

For a multigraded polynomial system \( \mathcal{F} \), the Dixon dialytic matrix is thus square, and its size is exactly the degree of the resultant of \( \mathcal{F} \). From the fact that the Dixon dialytic
matrix of a polynomial system $\mathcal{F}$ contains a multiple of its resultant, it follows that the determinant of this matrix gives exactly the resultant. We thus have:

**Theorem 7.1.** Given an unmixed generic multigraded system $\mathcal{F}$, the determinant of its Dixon dialytic matrix is exactly the resultant of $\mathcal{F}$. Moreover, depending on the order of the variable blocks, there exist $r!$ such Sylvester-type matrices.

In Sturmfels and Zelevinski (1994) it has been shown that for multigraded systems of type $(l_1, l_2, \ldots, l_r; k_1, k_2, \ldots, k_r)$, the multi-index $(m_1, \ldots, m_r)$, where $m_i = (k_i - 1)l_i + k_i \sum_{j=1}^{i-1} l_j$, constitutes the multiplier sets for constructing a square Sylvester-type resultant matrix whose determinant is non-zero for generic coefficients. For generic multigraded systems, the multiplier set used for each polynomial in the Dixon dialytic matrix construction is precisely the same as the multi-index in Sturmfels and Zelevinski (1994).

The Dixon dialytic construction results in exact Sylvester-type resultant matrices not only for generic unmixed multigraded systems but also for a much wider class of polynomial systems, without any prior knowledge about the structure of the polynomial systems. This is in contrast to the construction proposed in Sturmfels and Zelevinski (1994) which only applies for unmixed generic multigraded systems.

8. Conclusions

For multivariate polynomial systems, a new algorithm for constructing Sylvester-type matrices, called the Dixon dialytic matrices, is introduced, based on the Dixon formulation, for simultaneously eliminating many variables. The resulting matrices are sparse; i.e., their size is determined by the supports of the polynomials in a polynomial system. However, unlike other algorithms for constructing sparse resultant matrices which explicitly use the support structure of the polynomial system in the construction, the proposed algorithm exploits the sparse structure only implicitly, just like the generalized Dixon matrix construction.

The algorithm uses an arbitrary term to construct the multiplier sets for each polynomial in the polynomial system. In the unmixed case, it is shown that the Dixon dialytic matrices are of the smallest size if the term used in the construction is from the support. The size of a Dixon dialytic matrix puts an upper bound on the degree of the projection operator that can be extracted from it, thus also determining an upper bound on the degree of the extraneous factor, in the case where the projection operator is not exactly the resultant. Consequently, the degree of extraneous factors can be minimized by minimizing the size of the associated Dixon dialytic matrices. It is also shown that it does not matter what term is picked from the support for unmixed cases.

It is also shown that for generic multigraded polynomial systems, the Dixon dialytic matrices constructed using the proposed method are exact in the sense that their determinant is the resultant of the polynomial system.

In the mixed case, however, the choice of a term for construction becomes crucial. It is shown that if a term is selected from a non-empty intersection of the supports of the polynomials in a polynomial system, then the resulting Dixon dialytic matrices are
smaller. Since translation of supports does not affect the size of the Dixon dialytic matrices, translating supports so as to maximize the overlap among translated supports and selecting a term from this overlap can be formulated as an optimization problem.

The new method is compared theoretically and empirically with other methods for generating Sylvester-type resultant matrices, including subdivision and incremental algorithms for constructing sparse resultants proposed in Canny and Emiris (2000). These results theoretically confirm the practical advantages of the Dixon resultant formulation, which have been observed in a number of applications.

Acknowledgements

This research is supported in part by NSF grant Nos CCR-0203051, CDA-9503064, a grant from the Computer Science Research Institute at Sandia National Laboratories and the Computing and Information Technology Center (CITeC) at the University of Texas–Pan American.

References


