Behavioral effects of problem structure in isomorphic problem solving situations

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The behavior of subjects solving a problem is described by paths through the problem's state space representation. It is assumed that structural features of a particular problem — especially its possible subproblem and symmetry decompositions — effect the behavior or problem solving subjects. Furthermore, these patterns of behavior will be independent of the solver's individual strategies or heuristics. The theoretical expectations and empirical results of subjects solving the Tower of Hanoi and Tea Ceremony problems are presented, and the notion of "transfer" is discussed in the context of problems of related structure.

1. INTRODUCTION

Several recent research reports studying information processing in humans have analyzed the strategy/behavior distinction in problem solving. Dienes & Jeeves [5] and Branca & Kilpatrick [4] considered subjects solving card tasks structured on the klien group and cyclic group of order four. The studies compared subjects' behaviors on these tasks with their retrospective (after the fact) accounts of their strategies.

More recently Greeno [6], Thomas [15], and Reed, Ernest & Banerji [13] have studied solutions of subjects solving various forms of the famous Missionaries and Cannibals problem. In these studies the primary interest was focused on the sequences of moves (behaviors) subjects made in playing the games rather than on the specific strategies or heuristics they might have devised to determine their moves. Indeed, most current research acknowledges as fact that problem-solving strategies do not generate unique behavior, that is, a single pattern of behavior may be the result of the application of any of several different rules or strategies.

The state space analysis of problem solving behavior proposed in earlier work by the author [9] allows the strategy/behavior analysis to be even more precise. The *state space representation* of a problem [12] is the set of distinguishable configurations or situations of a problem together with the set of permitted moves or steps from one problem situation to another. Thus the state space representation — expressed as a directed graph — consists of an initial state together with all the states that may be reached from the initial state by the successive legal moves of the problem. One or more of these successor states are classified as goal states. If a problem's description clearly specifies the initial state, goal state(s) and set of legal moves of the problem, then its state space representation will be unique.

Besides Nilsson, Banerji [1], and Banerji & Ernest[2] have offered mathematical descriptions to characterize state spaces. This "state space algebra"

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allows such concepts as problem comparison, decomposition, and extension to be well defined, and also allows problem solving studies in the area of problem analogy, transfer, and generalization to be extremely precise.

A second use of the state space representation is to allow analysis of problem solving behavior in light of the fixed symmetry and subproblem structure of the problem[9]. Behaviors are recorded (by the researcher) as paths through the state space representation of the problem corresponding to the steps taken or moves made by each subject solving the problem. It is not suggested in this analysis that the solver in any way "perceives" the state space as an entity during problem solving. Rather, the symmetry properties and subproblem decompositions of the problem are formal properties of the state space which may or may not correspond to the geometrical or other perceptual properties of the problem readily apparent to the problem solver.

Hypotheses may then be formulated to predict the effects of problem structure on problem solving behavior. That is, for a fixed population of subjects and a fixed problem, hypotheses predicting patterns in the subjects' behavior (paths through the state space) can be formulated and tested. As stated above, these hypotheses may be tested regardless of the specific strategy the problem solver employs.

In the next section general hypotheses regarding the effects of problem structure on the behavior of problem-solving subjects are given. Three studies analysing this behavior are introduced, and the problems used in the study described.

II. GENERAL HYPOTHESES

In solving problems of fixed structure (a unique state space) the following hypotheses of a more or less general nature are suggested.

Hypothesis 1. (a) In solving a problem or subproblem the subject generates non-random, goal-directed paths in the state space representation of the problem or subproblem, and (b) when sub-goal states are entered the path exits from the respective subproblem.

Hypothesis 2. Identifiable "episodes" occur during problem solving corresponding to the solution of various subproblems. That is, path segments occur during problem solving which do not constitute the solution of a problem, but which do make up the solution of subproblems of the problem.

Hypothesis 3. The problem solver's paths through problems of identical (isomorphic) structure tend to be congruent.

Hypothesus 4. Given a symmetry within the state space of the problem, subjects exhibit succesive path segments in the state space congruent *modulo* this symmetry. (Goldin & Luger[7], describe the symmetries of a problem as the group of automorphisms of the problem's state space onto itself).

Hypothesis 5. Given two problems of similar structure, the effects of solving one problem will be reflected in the behavior evidenced in solving the second problem.

It may be that the validity of hypotheses 1 and 2 depends on the particular way that the state space of the problem is decomposed into subproblems since such a decomposition is not always unique. Hypothesis 4 (symmetry acquisition) is suggestive of the "insight" phenomenon which changes the gestalt of the problem solver [16] and often plays an important role in the eventual problem solution.

These general hypotheses have been tested in three specific problem situations. In the first study 51 subjects solved the Tower of Hanoi problem [9]. In the second study 21 subjects solved the Tea Ceremony problem (A Tower of Hanoi isomorph suggested by Newell); and in the third study 43 subjects solved both problems in a test for transfer.

In the Tower of Hanoi problem four concentric rings (labelled 1, 2, 3, 4 respectively) are placed in order of size, the largest on the bottom, on the first of three pegs (labelled A, B, C); the apparatus is pictured in Figure 1. The object of the problem is to transfer all the rings from peg A to peg C in the minimum number of moves. Only one ring may be moved at a time, and no larger ring may be placed over a smaller one on any peg.

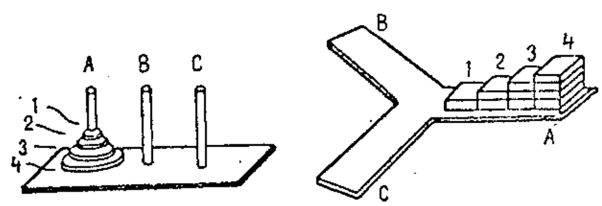

FIGURE 1. The 4-ring Tower of Hanoi Board and the 4-task Tea Ceremony problem in them "Start" states.

The Tea Ceremony, see Figure 1, is an isomorph of the Tower of Hanoi. Three people — a host, an elder, and a youth — participate in the ceremony. There are four tasks they perform — listed in ascending order of importance: feeding the fire, serving cakes, serving tea, and reading poetry. The host performs all the tasks at the start of the ceremony, and the tasks are transferred back and forth among the participants until the youth performs all the tasks, at which time the ceremony is completed. There are two constraints on the one-at-a-time transfer of tasks: 1) only the least important task a person is performing may be taken, and 2) no person may accept a task unless it is less important than any task they perform at the time. The object of the Tea Ceremony game is to transfer all the four tasks from the host to the elder in the fewest number of moves.

In the isomorphic relationship between the Tea Ceremony and the Tower of Hanoi the people — host, youth, and elder — correspond respectively with pegs A, B, and C. The four tasks — feeding the fire, serving cakes, serving tea, and reading poetry — correspond respectively with rings 1,2, 3, and 4. It can be checked that the initial state, goal state, and legal moves of the two games correspond.

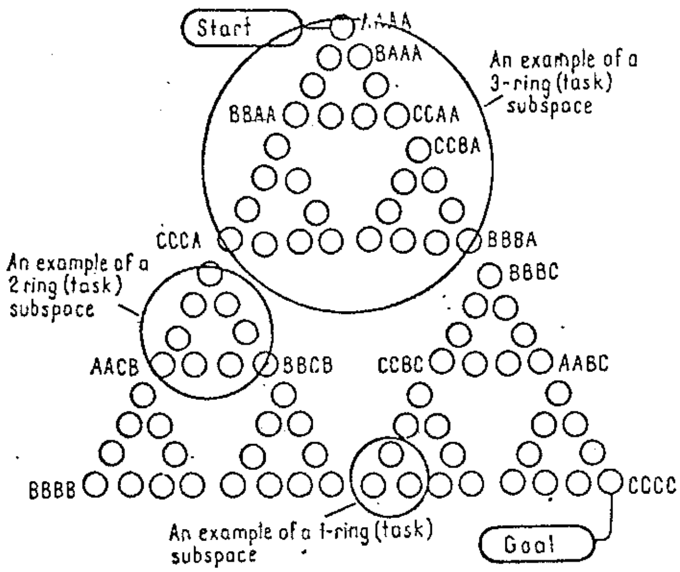

FIGURE 2. The State Space Representation of the Tower of Hanoi/Tea Ceremony problem. Legal moves effect transitions between adjacent states. Examples of subspaces are given.

Figure 2 is the complete state space representation of the Tower of Hanoi/Tea Ceremony problem. Each circle stands for a possible position or state of the games. The four letters labelling a state refer to the respective pegs (people) on which the four rings (tasks) are located. For example, state CCBC means that ring 1 (fire), ring 2 (cakes), and ring 4(poetry) are in their proper order on peg C (performed by the Elder). Ring 3 (tea) is on peg B (performed by the youth). A legal move by the problem solver always effects a transition between states represented by neighbouring circles in Figure 2. The solution path containing the minimum number of moves consists of the fifteen steps from AAAA to CCCC down the right side of the state space diagram.

The Tower of Hanoi/Tea Ceremony has a natural decomposition into nested subproblems. For example, to solve the 4-ring Tower of Hanoi problem, it is necessary at some point to move the largest ring from its original position on peg A to peg C, but before this can be done the three smaller rings must be assembled in their proper order on peg B. The problem of moving the three rings from one peg to another may be termed a 3-ring subproblem, and constitutes a subset of the state space of the 4-ring problem. The 4-ring state space contains three isomorphic 3-ring subspaces, for which the physical problem solving situations are different by reason of the position of ring 4. Each subspace becomes a suproblem when one of its entry states is designated as the initial state, and its exist states are designated as goal states. Similarly, each 3-ring subspace contains three isomorphic 2-ring subspaces for a total of nine in the 4-ring state space; and each 2-ring subspace may be further decomposed into three 1-ring subspaces, comprising only three states apiece. Note the examples in Figure 2 of 1-, 2-, and 3-ring subspaces.

Each n-ring subproblem, as well as the main problem, admits of a symmetry automorphism. The automorphism maps a goal state of the n-ring problem onto the conjugate goal state which corresponds to transferring the n-rings to the other open peg. Were the three pegs of the Tower of Hanoi board to be arranged at the corners of an equilateral triangle (as are the people in the Tea Ceremony), the symmetry automorphism would represent the geometric operation of reflection about the altitudes of the equilateral triangle.

In the next section hypotheses specific to the Tower of Hanoi and Tea Ceremony problems are tested and the results of these studies presented.

III. CRITERIA AND RESULTS

Criteria for satisfying the general hypotheses of section II were established as follows:

(1) a) The *non-randomness* of subjects paths was tested by comparing the number of "corners" or "turns" (as opposed to "straight" sections) in subjects paths with the number of "corners" or "turns" in paths of the same length generated randomly through the problem state space. Paths were *goal-directed* when they neither reentered any subproblem once it had been left nor, "moved away" from the problem or subproblem's goal state. A measurement of "movement away" from a goal state was possible by establishing a metric on the state space. This metric was the number of states in the shortest path between the subject's current state and the goal state.

b) The special role of subgoal states was investigated by determining the percentage of times that subjects' paths, once having reached the subproblem's goal state, immediately exited from that subproblem space. This percentage was compared with the percentage based on random choice of possible exits from the subgoal state (50 percent).

(2) An *n-ring episode* in solving the Tower of Hanoi problem was defined to occur when a subject executed minimal solutions to 50 percent or more of all the n-ring subproblems prior to executing minimal solutions to 50 percent or more of the n + 1-ring subproblems. Three such episodes were theoretically possible in solving the 4-ring Tower of Hanoi problem. It was asked whether a significant number of subjects would evidence at least one of these episodes in their solution, and whether any subjects would evidence all three.

(3) A subject's paths tended to be congruent in n-ring subproblems if any one congruence class of non-minimal solution paths predominanted in frequency, i.e. contained 60% more paths than the other non-minimal classes of paths. (Only paths on the 2- and 3-ring levels were considered).

(4) A subject exhibited symmetry when an interruption in a path occurred, followed immediately by the subject's formation of the symmetric image of the interrupted path. *A priori* it was predicted that such interruptions would occur for half the subjects, since probability dictates that 50 percent of the subjects' paths would start towards the goal state, and 50 percent in the symmetrically opposite direction.

(5) Transfer was tested by comparing (via the t-test) the amount of time used and number of states entered by subjects solving each of the problems.

It should be evident that these studies do not rely solely on conventional statistical tests for establishing the existence of effects in an population of subjects; rather, they suggest some new techniques for establishing the existence of "patterns" in subjects' behavior. For example, it seemed natural to consider "local" properties of the path — non-random, goal-directed paths were thought to have less "corners" or "turns", less "loops", and to have less "wandering about" that random, undirected paths. The above criteria attempted to make these notions more concrete: Using a "metric" on the state space, analysis of congruence of path segments, and the interruptions in paths are techniques used to establish the existence of patterns in subjects behavior.

A. The Tower of Hanoi Study. In the Tower of Hanoi Study hypotheses (1), (2), (3), and (4) above were tested on 51 college educated adults.[9] These subjects had no prior acquaintance with the Tower of Hanoi problem. Once having started the problem, the subject continued to work on it until he or she either gave up or succeeded in moving all the rings from the start to the goal peg in the least possible number of moves. The subject could start the problem over at any time or for any reason he or she wished. The total time spent was usually 15 to 20 minutes. A tape recorder was kept running continously to record the sequence of moves and any verbal responses of the subject.

Hypoteshis 1. a) In 45 (6 of the 51 subjects solved the problem on their first trial) subjects' first trials at solving the problem 95% met the criterion for nonrandomness. That is, subjects' first attempt at solving the problem deviated from paths randomly drawn through the Tower of Hanoi state space by more than one standard deviation in the occurrence of "corners" in the paths. 78% deviated from the random by more than two standard deviations. Of all 131 trials by subjects, 97% met the criterion for nonrandomness and 81% deviated from the random by more than two standard deviations. All deviations were in the direction of *fewer* "corners" in the paths. Of 45 subjects first attempts, 87% satisfied the criterion for goal-directedness; and 93%

of the subjects' 133 total attempts satisfied the criterion. Of the 685 paths through 2-ring subproblems, 96% met the criterion for subgoal directedness. Of the 321 paths through 3-ring subproblems, 91% met the criterion.
b) Of the 685 paths through 2-ring subproblems, 96% met the exist criterion; of the 321 3-ring subproblem paths, 98% met the exist criterion.

In figure 3, all paths of the subject deviated from random paths in respect to the number of "corners" by more than 2 standard deviations. (The

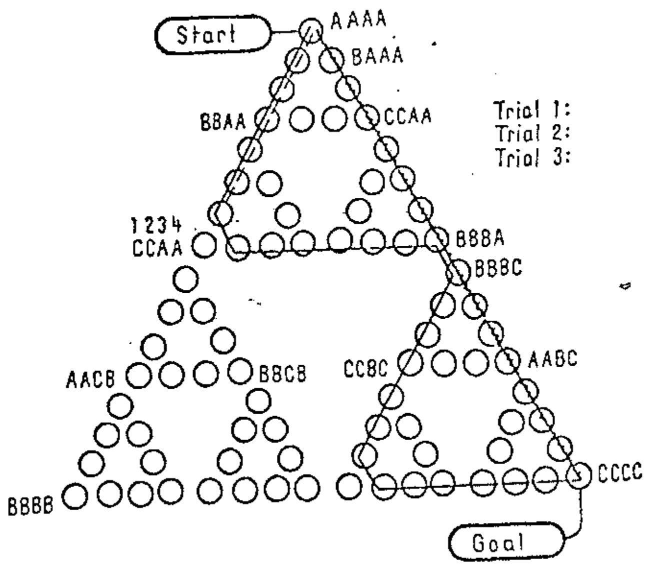

FIGURE 3. The behavior of one subject solving the Tower of Hanoi problem

number of corners per state entered by a random path was 67; s.d. $\pm$ 10). All paths of this subject were both goal and subgoal directed. All paths of this subject existed from the subproblem space once a subgoal state was entered. (The exit of a random path was 50%).

Hypothesis 2. A maximum of three "episodes" were possible, corresponding to solutions of 1-, 2-, and 3-ring subproblems respectively. Of all 51 subjects 45 (88%) displayed at least one of these "episodes"; 16 (31%) displayed just one; 22 (43%) displayed exactly two "episodes"; and all three theoretically possible "episodes" were displayed by 7 subjects (14%). As examples of episodes' consider the subject of Figure 3:

(1 = minimal and 0 = non-minimal subproblem paths)

Note that 2-, 3-, and 4-ring (sub) problems are placerd over each other in such a manner as to indicate the time sequence (left to right) of problem spaces entered.

The sequence of subproblems entered by the subject:

	2-ring episode		3-ring episode	
2-ring	1 0 1	1 0 1	1 1	1 1 1 1
3-ring	0	0	1	1 1
4-ring	0		0	1

There was no 1-ring episode. During the 2-ring episode 67% of the 2-ring subproblems were solved minimally and 0% of the 3-ring. During the 3-ring episode 100% of the 3-ring were solved minimally and 0% of the 4-ring.

Hypothesis 3. For the 2-ring subproblem, predominance of one congruence class of non-minimal path occurred for 7 of 45 subjects (16%); non-predominance occurred for 6 subjects (14%); and 32 subjects (70%) permitted no conclusion to be reached (because of insufficiently many nonminimal path, or an inconclusive distribution of these paths). For the 3-ring subproblem, predominance of one congruence class was shown by 6 subjects (13%); non-predominance by 21 subjects (41%); and 18 subjects (40%) permitted no conclusion. In trial 1 of Figure 3 there are congruent non-minimal solution paths through 2-ring subproblems (the second and fifth entered) and also through 3-ring subproblems (the first and second entered). In general, due to an insufficient number of non-minimal paths and an inconclusive distribution of these paths, the expected congruence of non-minimal solution paths across isomorphic subproblems was *not* confirmed by the data.

Hypothesis 4. Of 45 subjects, 44% displayed the predicted effect of the problem symmetry, by producing consecutive path segments congruent *modulo* the symmetry transformation of the problem. 7% exhibited this pattern two or more times during the problem solving. This compares reasonably well with the figure of 50% for whom the phenomena was predicted to occur. In figure 3, the subject displayed this effect in the second and third trials.

B. The Tea Ceremony. In this study hypotheses (1), (2), (3), and (4) above were tested with 21 adult subjects. The testing procedure was identical and the results were similar to those of the previous study.

Hypothesis 1. a) The subjects' paths were both non-random and goal-directed. b) Of 418 paths through 2-ring subproblems, 95.2% met the subgoal exit criterion; of 214 paths through 3-ring subproblems, 98.6% met the subgoal exit criterion. In figure 4, the behavior of one subject solving the Tea Ceremony problem, all paths were both non-random and goal directed; there were no violations of the subproblem exit criterion.

Hypothesis 2. All 21 subjects displayed at least one "episode", with 24% displaying exactly one episode. 52% displaying exactly two episodes, and 24%

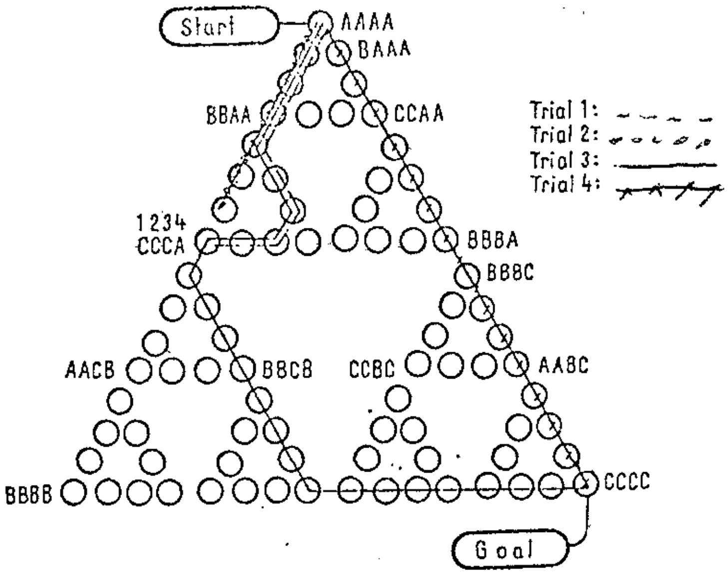

FIGURE 4. The behavior of one subject solving the Tea Ceremony problem.

displaying all three theoretically possible episodes. Consider the subject in Figure 4:

(1 = minimal, and 0 = non-minimal subproblem paths)

	2-ring episode	3-ring episode					
2-ring	1 0	1 1	1 1	1 0	1 1	1 1	1 1
3-ring	0	1	1	0	1	1	1
4-ring	0	0		0		1	

No 1-ring episode. In the 2-ring episode 50% of the 2-ring subproblems were solved minimally and 0% of the 3-ring. In the 3-ring episode 75% of the 3-ring subproblems were solved minimally and 0% of the 4-ring.

Hypothesis 3. Although the subject in Figure 4 showed a predominance of non-minimal 2-ring paths (the second and eighth) and of 3-ring paths (the first and fourth), this hypothesis was not verified across the population of subjects. This was due both to an insufficient number of non-minimal paths and an inconclusive distribution of these paths.

Hypothesis 4. Of 21 subjects, 14 (67%) displayed the effects of problem symmetry, and 4 subjects (19%) displayed this phenomenon more than once. The subject in Figure 4 had an interrupted path and production of a symmetrically conjugate path in the third and fourth trials.

C. The Transfer Study. The transfer study was recently designed and only preliminary results are presented. When the study is complete the effects on transfer of the subproblem and symmetry structure of the problem will be more fully delineated, i.e., hypotheses (1 b), 2), (3), and (4) above will be analysed for transfer effects. At this time only hypothesis 5 is considered.

Hypothesis 5. 43 subjects solved both the Tower of Hanoi and Tea Ceremony problems. 24 solved the Tower of Hanoi first and then the Tea Ceremony and 19 solved the problems in the reversed order. The mean time (in seconds) and mean number of states entered are given for each problem. The results of the "t" test are also given. (σ = one standard deviation).

Mean Time Used (seconds)

Group 1 (n = 19)	TC 518 (σ = 366)	TOH 149 (σ = 125)
Group 2 (n = 23)	TOH 399 (σ = 210)	TC 306 (σ = 223)

t on TOH = 4.54, significant at .01
t on TC = 2.32, significant at .05

Mean Number of States Entered

Group 1 (n = 19)	TC 101 (σ 60)	TOH 40 (σ = 29)
Group 2 (n = 23)	TOH 75 (σ = 33.5)	TC 66 (σ = 33.5)

t on TOH = 3.55, significant at .01
t on TC = 2.42, significant at .02

IV. SUMMARY AND CONCLUSIONS

The author has attempted to define clearly the strategy/behavior relationship and to establish a framework for studying the effects of problem structure on problem solving behavior. The state space representation of a problem is used to describe a problem's structure formally. This formalization allows comparison between problems and subproblems of related structure, and provides a precise description of the symmetry and subproblem decompositions of a problem, and thus permits the study of the effects of the structure of a problem on the behavior of subjects solving the problem. The behavior of subjects are recorded as paths through the state space according to the steps taken or moves made by the subject.

The results of experiments based on this method seem to confirm an important role played by features of the problem structure in determining patterns in the problem solving behavior of subjects. In particular:

— 1. the goal-directed behavior within subproblems and immediate exit from the subproblem space once the subproblem's goal state was achieved, indicates the problem solver's effective "decomposition" of the problem in attempting its solution;

— 2. the "episodes" within the problem's solution indicate the effect on the problem solver of the structure of the subproblems;

— 3. these episodes also seem to indicate, at least in the context of the Tower of Hanoi and Tea Ceremony, the problem's solution is found in a "bottom up" progression with smaller units solved throughout the problem before larger units. (It is interesting to compare these facets of humans' solution to the Tower of Hanoi problem with the mechanical solutions generated by the General Problem Solver [6, 11]);

— 4. the symmetry structure within the problem was reflected in the problem solver's interrupted paths;

— 5. finally, using the state space, transfer effects across problems of related structure can be analysed.

In concluding, it should be noted that the suggested general hypotheses of § II were only tested over a single problem structure and with a limited population of subjects — each of these hypotheses ought to be tested in different problem solving situations. "Transfer" between problems of related structure should be further considered, and in particular, the effects on transfer of the subproblem and symmetry structures of the problems should be more closely investigated. The author is currently collecting and further analysing data for the transfer study[3].

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