YET ANOTHER SINGLE LAW FOR LATTICES

WILLIAM MCCUNE, R. PADMANABHAN, AND ROBERT VEROFF

ABSTRACT. In this note we show that the equational theory of all lattices is defined by the single absorption law

$$(((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (w \vee ((s \vee x) \wedge (x \vee t))) = x.$$

This identity of length 29 with 8 variables is shorter than previously known such equations defining lattices.

1. INTRODUCTION

Given a finitely based equational theory of algebras, it is natural to determine the least number of equations needed to define that theory. Researchers have known for a long time that all finitely based group theories are one based [2]. Because of lack of a cancellation law in lattices (i.e., the absence of some kind of a subtraction operation), it was widely believed that the equational theory of lattices cannot be defined by a single identity. This belief was further strengthened by the fact that two closely related varieties, semilattices and distributive lattices, were shown to be not one based [9, 7].

In the late 1960s researchers attempted to formally prove that lattice theory is not one based by trying to show that no set of absorption laws valid in lattices can capture associativity. Given one such attempt [6], Padmanabhan pointed out that Sholander's 2-basis for distributive lattices [10] caused the method to collapse. This failure of the method led to a proof of existence of a single identity for lattices [7]. For a partial history of various single identities defining lattices, their respective lengths, and so forth, see the latest book on lattice theory by G. Grätzer [1, p. 477].

In this paper we present an absorption law of length 29 with 8 variables that characterizes the equational theory of all lattices. To the best of our knowledge, the shortest previously known single identity has length 77 with 8 variables [11]. Table 1 summarizes the results.

2. METHODOLOGY

The previously known single identities for lattice theory were found by procedures that take a basis in the form of absorption equations and reduce the size of the basis to a single equation. Such procedures typically produce very large equations.

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TABLE 1. Discovery of Single Identities for Lattice Theory

Authors	Reference	Length	Variables
R. McKenzie	1970 [7]	300,000	34
R. Padmanabhan	1977 [8]	243	7
W. McCune, R. Padmanabhan	1996 [5]	79	7
R. Veroff	2001 [11]	77	8
W. McCune, R. Padmanabhan, R. Veroff	2002	29	8

The single identities presented here were found by enumerating lattice identities, filtering them through sets of nonlattices, then trying to prove automatically that the surviving equations are single identities. Several programs were used.

Eq-enum: enumerates equations of the form $\alpha = x$, where α is in terms of meet, join, and variables. Each variable in α, aside from x, has one occurrence (most-general absorption lattice identities have this property), and neither the leftmost nor the rightmost variable in α is x (such identities are eliminated by projection models).

Lattice-filter: takes a stream of equations, uses Whitman's algorithm to decide which are lattice identities, and discards the nonidentities.

SEM [12] **and MACE** [4]**:** search for small, finite nonlattice models of equations.

Model-filter: takes a set of finite structures (nonlattices in our case) and a stream of equations and discards equations that are true in any of the structures. Most of the nonlattices were found by SEM or MACE; several were constructed by hand while examining candidate identities.

OTTER [3]**:** searches for proofs of first order and equational statements. In this case, it took candidate identities and tried for several seconds to prove basic lattice properties such as commutativity, associativity, idempotence, and absorption identities.

The general method was to apply the preceding programs and incrementally build a set of nonlattice structures by using SEM and MACE to search for nonlattice models of the current candidate. Some of these structures were then added to the set and used for filtering the subsequent candidates.

The programs were combined into a single program that was driven by the choice of which equations to enumerate (for example length 25 with 7 variables), and which nonlattices to use for filtering. It was run on several hundred processors, usually in jobs of 10–20 hours, over a period of several weeks. If the computation had been done on one processor, it would have taken several years. About half a trillion equations were enumerated, and the set of nonlattices grew to be several thousand, most of size 4. About 100,000 candidates survived the model filter program, and OTTER proved basic lattice properties for several hundred of those. Further OTTER searches on those candidates showed the following two to be single identities.

(A1) $$(((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (w \vee ((s \vee x) \wedge (x \vee t))) = x$$

(A2) $$(((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (((w \vee x) \wedge (s \vee x)) \vee t) = x$$

The first proof that (A1) is a basis took OTTER several minutes and was more than 250 steps. The standard lattice theory 6-basis (commutativity, associativity, absorption) was derived. We then had OTTER prove McKenzie's 4-basis (given

in the next section), which produced a proof of about 170 steps in less than one minute. The 50-step proof given below was produced by Larry Wos, who used various methods to simplify OTTER's proof of the McKenzie basis.

The search for single identities was not complete; that is, many shorter equations and equations with fewer variables were considered for which we could find neither countermodels nor proofs. Therefore, whether there exists a shorter single identity is an open question.

3. Proof

McKenzie's well-known 4-basis for lattices consists of the following equations.

(L1) $$x \vee (y \wedge (x \wedge z)) = x$$

(L2) $$x \wedge (y \vee (x \vee z)) = x$$

(L3) $$((y \wedge x) \vee (x \wedge z)) \vee x = x$$

(L4) $$((y \vee x) \wedge (x \vee z)) \wedge x = x$$

The following (machine-oriented) proof is a derivation of {L1,L2,L3,L4} from (A1). If i stands for the equation $u_i = v_i$, and j stands for $u_j = v_j$, then $[i \to j]$ justifies an equation $u = v$ obtained in the following way. Take a subterm s of u_j such that u_i and s are unifiable by substitutions f and g; that is, $f(u_i)$ and $g(s)$ are identical. Let u be the term $g(u_j)$, but with an occurrence of $g(s)$ replaced by $f(v_i)$, and let v be the term $g(v_j)$.

1 $(((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (w \vee ((s \vee x) \wedge (x \vee t))) = x$ [A1]

2 $(((x \vee y) \wedge y) \vee (y \vee y)) \wedge (z \vee ((u \vee y) \wedge (y \vee v))) = y$ [1 → 1]

3 $(((x \vee (y \vee y)) \wedge (y \vee y)) \vee ((y \vee y) \vee (y \vee y))) \wedge (z \vee y) = y \vee y$ [2 → 2]

4 $(((x \vee y) \wedge y) \vee (((y \vee y) \vee (z \wedge y)) \wedge u)) \wedge (v \vee ((w \vee y) \wedge (y \vee t))) = y$ [3 → 1]

5 $(((x \vee (((y \vee y) \vee (z \wedge y)) \wedge u)) \wedge (((y \vee y) \vee (z \wedge y)) \wedge u)) \vee ((((y \vee y) \vee (z \wedge y)) \wedge u) \vee$
$(((y \vee y) \vee (z \wedge y)) \wedge u))) \wedge (v \vee y) = ((y \vee y) \vee (z \wedge y)) \wedge u$ [4 → 2]

6 $(((x \vee y) \wedge y) \vee (((((y \vee y) \vee (z \wedge y)) \wedge u) \vee (v \wedge y)) \wedge w)) \wedge (t \vee ((s \vee y) \wedge (y \vee r))) = y$ [5 → 1]

7 $(((x \vee y) \wedge y) \vee (z \wedge y)) \wedge (u \vee ((v \vee y) \wedge (y \vee w))) = y$ [6 → 6]

8 $(((x \vee (y \wedge z)) \wedge (y \wedge z)) \vee (u \wedge (y \wedge z))) \wedge (v \vee z) = y \wedge z$ [7 → 7]

9 $(((x \vee y) \wedge y) \vee (((z \wedge y) \vee (u \wedge y)) \wedge v)) \wedge (w \vee ((t \vee y) \wedge (y \vee s))) = y$ [8 → 1]

10 $(((x \vee y) \wedge y) \vee y) \wedge (z \vee ((u \vee y) \wedge (y \vee v))) = y$ [9 → 9]

11 $(((x \vee y) \wedge y) \vee (((z \wedge y) \vee (u \wedge y)) \wedge v)) \wedge (w \vee y) = y$ [10 → 9]

12 $(((x \vee y) \wedge y) \vee (z \wedge y)) \wedge (u \vee y) = y$ [10 → 7]

13 $(((x \vee y) \wedge y) \vee (y \vee y)) \wedge (z \vee y) = y$ [10 → 2]

14 $(x \vee (y \wedge (x \vee x))) \wedge (z \vee ((u \vee (x \vee x)) \wedge ((x \vee x) \vee v))) = x \vee x$ [13 → 7]

15 $(x \vee x) \wedge (y \vee ((z \vee (x \vee x)) \wedge ((x \vee x) \vee u))) = x \vee x$ [13 → 14]

16 $(((x \vee y) \wedge y) \vee ((z \wedge y) \vee (z \wedge y))) \wedge (u \vee y) = y$ [15 → 11]

17 $((x \wedge y) \vee (x \wedge y)) \wedge (z \vee y) = (x \wedge y) \vee (x \wedge y)$ [16 → 15]

18 $((x \vee y) \wedge y) \vee ((x \vee y) \wedge y) = y$ [12 → 17]

19 $(x \wedge y) \wedge (z \vee y) = x \wedge y$ [18 → 8]

20 $x \wedge (y \vee x) = x$ [18 → 12]

21 $x \wedge (y \vee ((z \vee x) \wedge (x \vee u))) = x$ [18 → 7]

22 $(((x \vee y) \wedge y) \vee ((z \wedge y) \vee (u \wedge y))) \wedge (v \vee y) = y$ [20 → 11]

23 $((x \wedge y) \vee (z \wedge y)) \wedge (u \vee y) = (x \wedge y) \vee (z \wedge y)$ [22 → 21]

24 $((x \vee y) \wedge y) \vee (z \wedge y) = y$ [12 → 23]

25 $x \wedge (x \vee y) = x$ [24 → 21]

26 $((x \vee y) \wedge y) \vee ((z \wedge y) \vee (u \wedge y)) = y$ [22 → 25]

27 $((x \vee y) \wedge y) \vee y = y$ [10 → 25]

28 $x \wedge (y \vee (x \vee z)) = x \wedge (x \vee z)$ [25 → 19]

29	$(x \wedge x) \vee x = x$		$[27 \to 27]$
30	$x \wedge (y \vee (x \wedge (x \vee z))) = x$		$[27 \to 21]$
31	$x \wedge ((y \vee x) \wedge (x \vee z)) = x$		$[27 \to 21]$
32	$x \wedge x = x$		$[27 \to 20]$
33	$x \wedge (y \vee (x \vee z)) = x$	(L2)	$[25 \to 28]$
34	$x \vee x = x$		$[32 \to 29]$
35	$(x \vee y) \wedge y = y$		$[24 \to 34]$
36	$x \wedge (y \wedge x) = y \wedge x$		$[24 \to 35]$
37	$(x \vee (((y \wedge x) \vee (z \wedge x)) \wedge u)) \wedge (v \vee x) = x$		$[35 \to 11]$
38	$((x \vee y) \wedge (y \vee z)) \wedge y = y \wedge ((x \vee y) \wedge (y \vee z))$		$[31 \to 36]$
39	$(x \vee (x \wedge y)) \wedge (z \vee x) = x$		$[24 \to 37]$
40	$x \vee (((y \wedge x) \vee (z \wedge x)) \wedge u) = x$		$[25 \to 37]$
41	$((x \vee y) \wedge (y \vee z)) \wedge y = y$	(L4)	$[31 \to 38]$
42	$(((x \wedge y) \vee (z \wedge y)) \vee (((x \wedge y) \vee (z \wedge y)) \wedge u)) \wedge y = (x \wedge y) \vee (z \wedge y)$		$[26 \to 39]$
43	$(x \wedge y) \wedge x = x \wedge y$		$[39 \to 31]$
44	$x \vee (((y \wedge x) \vee x) \wedge z) = x$		$[35 \to 40]$
45	$x \vee ((y \wedge x) \wedge z) = x$		$[40 \to 40]$
46	$((x \wedge y) \vee y) \wedge y = (x \wedge y) \vee y$		$[44 \to 30]$
47	$x \vee (y \wedge (z \wedge x)) = x$		$[36 \to 45]$
48	$(x \wedge y) \vee y = y$		$[35 \to 46]$
49	$x \vee (y \wedge (x \wedge z)) = x$	(L1)	$[43 \to 47]$
50	$((x \wedge y) \vee (z \wedge y)) \vee y = y$		$[42 \to 48]$
51	$((x \wedge y) \vee (y \wedge z)) \vee y = y$	(L3)	$[43 \to 50]$

Lines {49,33,51,41} are equations {L1,L2,L3,L4}, respectively. A similar OTTER proof shows that (A2) is a single identity.

The Web page

`www.mcs.anl.gov/~mccune/papers/ltsax`

contains several files associated with this note, including previously known single identities, OTTER input files that produced the proofs, and other supporting material.

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Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, Illinois 60419, U.S.A.
E-mail address: mccune@mcs.anl.gov
URL: http://www.mcs.anl.gov/~mccune

Department of Mathematics, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada
E-mail address: padman@cc.umanitoba.ca
URL: http://home.cc.umanitoba.ca/~padman

Department of Computer Science, University of New Mexico, Albuquerque, New Mexico 87131, U.S.A.
E-mail address: veroff@cs.unm.edu
URL: http://www.cs.unm.edu/~veroff