Lattice Laws Forcing Distributivity Under Unique Complementation

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Abstract

In this paper we give several new lattice identities valid in non-modular lattices such that a uniquely complemented lattice satisfying any of these identities is necessarily Boolean. Since some of these identities are consequences of modularity as well, these results generalize the classical result of Birkhoff and von Neumann that every uniquely complemented modular lattice is Boolean. In particular, every uniquely complemented lattice in $M \vee N_5$, the least non-modular variety, is Boolean.

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1 Introduction

In 1904 Huntington [4] conjectured that every uniquely complemented lattice must be distributive (and hence a Boolean algebra). In 1945, R. P. Dilworth shattered this conjecture by proving [2] that every lattice can be embedded in a uniquely complemented lattice. For a much powerful version of the same results, see Adams and Sichler [1].

In spite of these deep results, still it is hard to find "nice" examples of uniquely complemented lattices that are not Boolean. This is because uniquely complemented lattices having a little extra structure most often turn out to be distributive. This seems to be the essence of Huntington's conjecture. For example, we have the theorem of Garrett Birkhoff and von Neumann that every uniquely complemented modular lattice is Boolean. Following [8], we call a lattice property P a *Huntington property* if every uniquely complemented P-lattice is distributive. Similarly, a lattice variety K is said to be a *Huntington variety* if every uniquely complemented lattice in K is Boolean. In this terminology, the modular lattices are the largest known Huntington variety. A monograph by Salii [11] gives a comprehensive survey of known Huntington properties. Among these, modularity is the only known condition which is a lattice identity. In this paper, we give a number of new non-modular Huntington varieties and any of them could be construed as a generalization of von Neumann-Birkhoff theorem.

The automated theorem provers Otter [5] and Prover9 [7], and the program Mace4 [6], which searches for finite algebras, were used in this work. Several automated proofs are given in an appendix to this paper. The Web page associated with this paper [10] contains additional Huntington identities and automated proofs supporting this work.

2 A Non-modular Huntington Variety

Here we give a lattice identity that defines a non-modular Huntington variety. Several others are given in the following sections and in the supporting Web page [10].

Theorem 1. *The variety of lattices defined by*

$$(x \wedge (y \vee (x \wedge z))) \vee (x \wedge (z \vee (x \wedge y))) = x \wedge (z \vee y) \tag{H69}$$

is a non-modular Huntington variety.

Proof. We show that the condition $a \wedge b' = 0$ forces the inequality $a \leq b$ and hence by a well-known theorem of O. Frink [9], the lattice will necessarily be Boolean. Indeed, let $a \wedge b' = 0$ for some two elements a, b in a uniquely complemented lattice satisfying the identity

$$(x \wedge (y \vee (x \wedge z))) \vee (x \wedge (z \vee (x \wedge y))) = x \wedge (z \vee y).$$

Put $z = x'$ in the above to get

$$(x \wedge y) \vee (x \wedge (x' \vee (x \wedge y))) = x \wedge (x' \vee y).$$

Now let $x = b'$, $y = a$. We have

$$(b' \wedge a) \vee (b' \wedge (b \vee (b' \wedge a))) = b' \wedge (b \vee a).$$

So if we assume that $a \wedge b' = 0$, then we get $b' \wedge (b \vee a) = 0$. Also, $b' \vee (b \vee a) = (b' \vee b) \vee a = 1 \vee a = 1$. Thus both b and $b \vee a$ are complements of the element b'. Since the lattice is uniquely complemented, we get the desired conclusion $b \vee a = b$. In other words, we have proved that the given lattice satisfies the bi-implication $a \leq b$ if and only if $a \wedge b' = 0$ and hence, by Frink's theorem, the lattice is distributive. □

3 Huntington Implications

Here we show Huntington properties that are implications. These can be used, among other purposes, to show that lattice identities are Huntington.

Theorem 2. (See [8].) *A uniquely complemented lattice satisfying any one of the following three implications (or their duals) is distributive.*

$$x \vee y = x \vee z \Rightarrow x \vee y = x \vee (y \wedge z) \quad \text{(SD-}\vee\text{)}$$
$$x \vee y = x \vee z \Rightarrow (x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z) \quad \text{(CD-}\vee\text{)}$$
$$x \vee y = x \vee z \Rightarrow x \wedge ((x \wedge y) \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{(CM-}\vee\text{)}$$

A proof of (CD-$\vee$) is given in the appendix. Proofs of the other two cases are given on the supporting Web page [10].

Corollary 1. *A uniquely complemented lattice satisfying the identity*

$$x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (x \vee y))) = (x \wedge y) \vee (x \wedge z) \quad \text{(H82)}$$

is Huntington.

Proof. It is easy to see that (H82) implies the lattice implication (CD-$\vee$). Indeed, if

$$x \vee y = x \vee z,$$

then

$$\begin{aligned}(x \wedge y) \vee (x \wedge z) &= x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (xvy))) && \text{by (H82)}\\ &= x \wedge ((y \wedge (x \vee y)) \vee (z \wedge (x \vee z))) && \text{by hypothesis}\\ &= x \wedge (y \vee z)\end{aligned}$$

□

As the reader can see, the identity (H82) is designed to show that there are lattice identities which formally imply such implications. Using powerful concepts like the bounded homomorphisms of Ralph McKenzie, one could show that there many lattice identities like (H82) which formally imply (SD-$\vee$), (SD-$\wedge$), (CD-$\vee$), etc. In fact, every finite lattice satisfying (SD-$\vee$) or (SD-$\wedge$) will satisfy a lattice identity which formally implies the respective implication and all these identities are examples of non-modular Huntington identities (for more details, please see [8]).

Table 1 lists several Huntington identities justified by the preceding Huntington implications. Proofs can be found on the supporting Web page [10]. None of the identities are equivalent (given lattice theory).

Name	Identity	Reason
H18	$(x \wedge y) \vee (x \wedge z) = x \wedge ((x \wedge y) \vee ((x \wedge z) \vee (y \wedge (x \vee z))))$	CM-$\vee$
H50	$x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge (y \vee (z \wedge (x \vee (z \wedge (y \vee u)))))$	SD-$\vee$
H51	$x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge (y \vee ((x \wedge z) \vee (z \wedge u)))$	SD-$\vee$
H64	$x \wedge (y \vee z) = x \wedge (y \vee (x \wedge (z \vee (x \wedge (y \vee (x \wedge z))))))$	SD-$\wedge$
H68	$x \wedge (y \vee z) = x \wedge (y \vee (x \wedge (z \vee (x \wedge y))))$	SD-$\wedge$
H69	$x \wedge (y \vee z) = (x \wedge (z \vee (x \wedge y))) \vee (x \wedge (y \vee (x \wedge z)))$	SD-$\wedge$
H76	$x \wedge (y \vee (z \wedge (y \vee u))) = x \wedge (y \vee (z \wedge (u \vee (x \wedge y))))$	SD-$\vee$, SD-$\wedge$
H79	$x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge ((x \wedge (y \vee (x \wedge z))) \vee (z \wedge u))$	SD-$\vee$, SD-$\wedge$
H80	$(x \wedge y) \vee (x \wedge z) = x \wedge ((x \wedge y) \vee (z \wedge (x \vee (y \wedge (x \vee z)))))$	CM-$\vee$
H82	$(x \wedge y) \vee (x \wedge z) = x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (x \vee y)))$	CD-$\vee$, CM-$\vee$

Table 1: Huntington Identities Justified by Huntington Implications

4 More Huntington Identities

This section contains several non-modular Huntington identities that do not satisfy the Huntington implications (SD-$\vee$), (CD-$\vee$), (CM-$\vee$), or their duals.

Theorem 3. *The variety of lattices defined by*

$$x \wedge (y \vee z) = x \wedge (y \vee ((x \vee y) \wedge (z \vee (x \wedge y)))) \qquad \text{(H58)}$$

is a non-modular Huntington variety.

RP: do we need a reference that the order reversibility property is sufficient?

Proof. (The automatic proof from which this proof was derived is given in the Appendix.) We show that any uniquely complemented lattice satisfying (H58) also satisfies the order reversibility property $a \leq b \Rightarrow b' \leq a'$. Assume $a \leq b$ and therefore $a \wedge b' = 0$. In (H58) set $x = a, y = b, z = (a \wedge b')'$ and then simplify, giving

$$a \wedge (b' \vee (a \vee b')') = 0.$$

Then unique complementation gives $b' \vee (a \vee b')' = a'$ and therefore $b' \leq a'$. ☐

Additional Huntington identities not satisfying the Huntington implications are shown in the following list. Automated proofs are given on the supporting

Web page [10].

$$x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge (y \vee (z \wedge (x \vee (z \wedge u)))) \quad \text{(H1)}$$

$$x \wedge (y \vee (x \wedge z)) = x \wedge (y \vee (z \wedge ((x \wedge (y \vee z)) \vee (y \wedge z)))) \quad \text{(H2)}$$

$$x \wedge (y \vee (x \wedge z)) = x \wedge (y \vee (z \wedge (y \vee (x \wedge (z \vee (x \wedge y)))))) \quad \text{(H3)}$$

$$x \vee (y \wedge (x \vee z)) = x \vee (y \wedge (z \vee (x \wedge (z \vee y)))) \quad \text{(H55)}$$

$$x \wedge (y \vee z) = x \wedge (y \vee ((x \vee y) \wedge (z \vee (x \wedge y)))) \quad \text{(H58)}$$

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Appendix

Covers of N_5

We should include some introduction to these lattices.

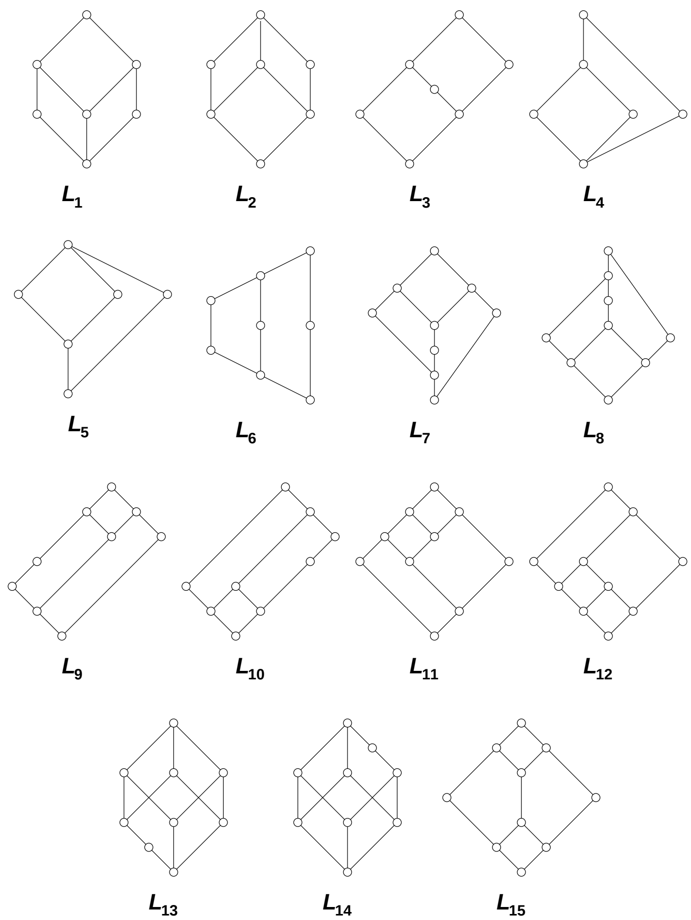

Figure 1: All Covers of the Least Non-Modular Lattice N_5

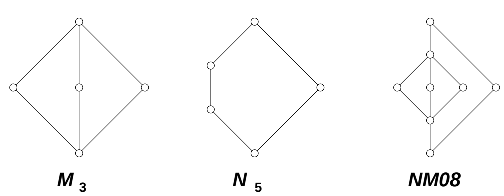

Figure 2: Lattices M_3, N_5, $NM08$

(H1)	1	2					7			10	11		13	14	15		N_5	
(H2)	1			4		6	7	8	9	10	11	12	13	14	15	M_3	N_5	$NM08$
(H3)	1			4		6	7	8	9	10	11	12	13	14	15	M_3	N_5	$NM08$
(H18)	1			4		6	7		9	10	11		13		15	M_3	N_5	$NM08$
(H50)	1					6	7		9	10	11		13	14	15		N_5	
(H51)	1						7			10	11		13	14	15		N_5	
(H55)		2			5	6	7	8	9	10		12	13	14	15	M_3	N_5	$NM08$
(H58)		2			5	6	7	8	9	10		12	13	14	15	M_3	N_5	$NM08$
(H64)		2				6	7	8	9	10	11	12	13	14	15		N_5	
(H68)		2				6	7	8	9	10		12	13	14	15		N_5	
(H69)		2				6	7	8	9	10		12	13	14	15		N_5	
(H76)						6	7	8	9	10			13	14	15		N_5	
(H79)							7			10			13	14	15		N_5	
(H80)	1		3	4		6	7	8	9	10	11		13		15	M_3	N_5	$NM08$
(H82)	1			4		6	7		9	10	11		13		15		N_5	

Table 2: Lattices ($L_1 - L_{15}$, M_3, N_5, $NM08$) for which the Identities Hold

Proof of Theorem 2, Part CD-$\vee$

This proof was produced by the program Prover9 [7]. The input and output files can be found on the supporting Web page [10].

13	$x \vee y = y \vee x$	[input]
14	$x \wedge y = y \wedge x$	[input]
15	$(x \vee y) \vee z = x \vee (y \vee z)$	[input]
16	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	[input]
17	$x \wedge (x \vee y) = x$	[input]
18	$x \vee (x \wedge y) = x$	[input]
19	$x \vee x' = 1$	[input]
20	$x \wedge x' = 0$	[input]
21	$x \vee y \neq 1 \ \mid\ x \wedge y \neq 0 \ \mid\ x' = y$	[input]
22	$A \wedge B = A$	[input]
23	$A' \vee B' \neq A'$	[input]
24	$x \vee y \neq x \vee z \ \mid\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	[input]
26	$x \wedge (y \wedge z) = y \wedge (x \wedge z)$	[14 → 16; 16]
32	$x \vee ((x \wedge y) \vee z) = x \vee z$	[18 → 15]
37	$x \vee (x' \vee y) = 1 \vee y$	[19 → 15]
39	$x \wedge 1 = x$	[19 → 17]
42	$x \vee 0 = x$	[20 → 18]
43	$A \wedge (B \wedge x) = A \wedge x$	[22 → 16]
51	$x \vee y \neq 1 \ \mid\ x \wedge (y \vee x') = 0 \vee (x \wedge y)$	[19 → 24; 20 13]
60	$1 \wedge x = x$	[39 → 14]
66	$0 \vee x = x$	[42 → 13]
69	$x \vee y \neq 1 \ \mid\ x \wedge (y \vee x') = x \wedge y$	[51; 66]
72	$1 \vee x = 1$	[60 → 17]
73	$x \vee (x' \vee y) = 1$	[37; 72]

75	$0 \wedge x = 0$	[66 → 17]
79	$x \wedge (y \wedge x') = y \wedge 0$	[20 → 26]
81	$x \vee 1 = 1$	[72 → 13]
83	$x \wedge 0 = 0$	[75 → 14]
84	$x \wedge (y \wedge x') = 0$	[79; 83]
103	$A \wedge (A' \vee B') \neq 0$	[21 a 73 a c 23 a]
170	$x \vee (x \wedge y)' = 1$	[19 → 32; 81]
185	$x \vee (y \wedge x)' = 1$	[14 → 170]
194	$B \vee A' = 1$	[22 → 185]
891	$B \wedge (A' \vee B') = B \wedge A'$	[69 a 194 a]
5852	$A \wedge (A' \vee B') = 0$	[891 → 43; 84]
5853	$\square$	[5852 a 103 a]

Proof of Theorem 3

This proof was produced by the program Prover9 [7]. The input and output files can be found on the supporting Web page [10].

29	$x \vee y = y \vee x$	[input]
37	$x \vee (y \wedge x) = x$	[input]
38	$x \vee (x \vee y) = x \vee y$	[input]
40	$x \wedge x' = 0$	[input]
47	$x \vee 0 = x$	[input]
48	$0 \vee x = x$	[input]
49	$x \vee y \neq 1 \;\mid\; x \wedge y \neq 0 \;\mid\; x' = y$	[input]
50	$x \wedge (x \vee y)' = 0$	[input]
51	$x \vee (y \vee (x \vee y)') = 1$	[input]
52	$x \wedge (y \vee ((x \vee y) \wedge (z \vee (x \wedge y)))) = x \wedge (y \vee z)$	[input]
53	$A \wedge B = A$	[input]
54	$A' \vee B' \neq A'$	[input]
103	$A \vee B = B$	[53 → 37; 29]
107	$A \wedge B' = 0$	[103 → 50]
109	$A \wedge (B' \vee ((A \vee B') \wedge x)) = A \wedge (B' \vee x)$	[107 → 52; 47]
2392	$A \wedge (B' \vee (A \vee B')') = 0$	[40 → 109; 29 48 107]
2439	$B' \vee (A \vee B')' = A'$	[49 a 51 a b 2392 a]
2481	$A' \vee B' = A'$	[2439 → 38; 29 2439]
2482	$\square$	[2481 a 54 a]