Lattice Laws Forcing Distributivity Under Unique Complementation

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Abstract

In this paper we give several new lattice identities valid in non-modular lattices such that a uniquely complemented lattice satisfying any of these identities is necessarily Boolean. Since some of these identities are consequences of modularity as well, these results generalize the classical result of Birkhoff and von Neumann that every uniquely complemented modular lattice is Boolean. In particular, every uniquely complemented lattice in $M \vee N_5$, the least non-modular variety, is Boolean.

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1 Introduction

In 1904 Huntington [4] conjectured that every uniquely complemented lattice must be distributive (and hence a Boolean algebra). In 1945, R. P. Dilworth shattered this conjecture by proving [2] that every lattice can be embedded in a uniquely complemented lattice. For a much powerful version of the same results, see Adams and Sichler [1].

In spite of these deep results, still it is hard to find "nice" examples of uniquely complemented lattices that are not Boolean. This is because uniquely complemented lattices having a little extra structure most often turn out to be distributive. This seems to be the essence of Huntington's conjecture. For example, we have the theorem of Garrett Birkhoff and von Neumann that every uniquely complemented modular lattice is Boolean. Following [10], we call a lattice property P a *Huntington property* if every uniquely complemented P-lattice is distributive. Similarly, a lattice variety K is said to be a *Huntington variety* if every uniquely complemented lattice in K is Boolean. In this terminology, the modular lattices are the largest previously known Huntington variety. A monograph by Salii [13] gives a comprehensive survey of known Huntington properties. Among these, modularity is the only known condition which is a lattice identity. In this paper, we give a number of new non-modular Huntington varieties and any of them could be construed as a generalization of the von Neumann-Birkhoff theorem.

The automated theorem provers Otter [6] and Prover9 [8], and the program Mace4 [7], which searches for finite algebras, were used in this work. Several automated proofs are given in an appendix to this paper. The Web page associated with this paper [12] contains additional Huntington identities and automated proofs supporting this work.

2 A Non-modular Huntington Variety

Here we give a lattice identity that defines a non-modular Huntington variety. Several others are given in the following sections and in the supporting Web page [12].

Theorem 1. *The variety of lattices defined by*

$$(x \wedge (y \vee (x \wedge z))) \vee (x \wedge (z \vee (x \wedge y))) = x \wedge (z \vee y) \tag{H69}$$

is a non-modular Huntington variety.

Proof. We show that the condition $a \wedge b' = 0$ forces the inequality $a \leq b$ and hence by a well-known theorem of O. Frink [11], the lattice will necessarily be Boolean. Indeed, let $a \wedge b' = 0$ for some two elements a, b in a uniquely complemented lattice satisfying the identity

$$(x \wedge (y \vee (x \wedge z))) \vee (x \wedge (z \vee (x \wedge y))) = x \wedge (z \vee y).$$

Put $z = x'$ in the above to get

$$(x \wedge y) \vee (x \wedge (x' \vee (x \wedge y))) = x \wedge (x' \vee y).$$

Now let $x = b'$, $y = a$. We have

$$(b' \wedge a) \vee (b' \wedge (b \vee (b' \wedge a))) = b' \wedge (b \vee a).$$

So if we assume that $a \wedge b' = 0$, then we get $b' \wedge (b \vee a) = 0$. Also, $b' \vee (b \vee a) = (b' \vee b) \vee a = 1 \vee a = 1$. Thus both b and $b \vee a$ are complements of the element b'. Since the lattice is uniquely complemented, we get the desired conclusion $b \vee a = b$. In other words, we have proved that the given lattice satisfies the bi-implication $a \leq b$ if and only if $a \wedge b' = 0$ and hence, by Frink's theorem, the lattice is distributive. □

3 Huntington Implications

Here we show Huntington properties that are implications. These can be used, among other purposes, to show that lattice identities are Huntington.

Theorem 2. (See [10].) *A uniquely complemented lattice satisfying any one of the following three implications (or their duals) is distributive.*

$$x \vee y = x \vee z \Rightarrow x \vee y = x \vee (y \wedge z) \qquad \text{(SD-}\vee\text{)}$$
$$x \vee y = x \vee z \Rightarrow (x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z) \qquad \text{(CD-}\vee\text{)}$$
$$x \vee y = x \vee z \Rightarrow x \wedge ((x \wedge y) \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \text{(CM-}\vee\text{)}$$

A proof of (CD-$\vee$) is given in the appendix. Proofs of the other two cases are given on the supporting Web page [12].

Corollary 1. *A uniquely complemented lattice satisfying the identity*

$$x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (x \vee y))) = (x \wedge y) \vee (x \wedge z) \qquad \text{(H82)}$$

is Huntington.

Proof. It is easy to see that (H82) implies the lattice implication (CD-$\vee$). Indeed, if

$$x \vee y = x \vee z,$$

then

$$\begin{aligned}(x \wedge y) \vee (x \wedge z) &= x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (xvy))) && \text{by (H82)}\\ &= x \wedge ((y \wedge (x \vee y)) \vee (z \wedge (x \vee z))) && \text{by hypothesis}\\ &= x \wedge (y \vee z)\end{aligned}$$

□

As the reader can see, the identity (H82) is designed to show that there are lattice identities which formally imply such implications. Using powerful concepts like the bounded homomorphisms of Ralph McKenzie, one could show that there are many lattice identities like (H82) which formally imply (SD-$\vee$), (SD-$\wedge$), (CD-$\vee$), etc. In fact, every finite lattice satisfying (SD-$\vee$) or (SD-$\wedge$) will satisfy a lattice identity which formally implies the respective implication and all these identities are examples of non-modular Huntington identities (for more details, please see [10]).

Table 1 lists several Huntington identities justified by the preceding Huntington implications. Proofs can be found on the supporting Web page [12]. None of the identities are equivalent (given lattice theory).

Name	Identity	Reason
H18	$(x \wedge y) \vee (x \wedge z) = x \wedge ((x \wedge y) \vee ((x \wedge z) \vee (y \wedge (x \vee z))))$	CM-$\vee$
H50	$x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge (y \vee (z \wedge (x \vee (z \wedge (y \vee u)))))$	SD-$\vee$
H51	$x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge (y \vee ((x \wedge z) \vee (z \wedge u)))$	SD-$\vee$
H64	$x \wedge (y \vee z) = x \wedge (y \vee (x \wedge (z \vee (x \wedge (y \vee (x \wedge z))))))$	SD-$\wedge$
H68	$x \wedge (y \vee z) = x \wedge (y \vee (x \wedge (z \vee (x \wedge y))))$	SD-$\wedge$
H69	$x \wedge (y \vee z) = (x \wedge (z \vee (x \wedge y))) \vee (x \wedge (y \vee (x \wedge z)))$	SD-$\wedge$
H76	$x \wedge (y \vee (z \wedge (y \vee u))) = x \wedge (y \vee (z \wedge (u \vee (x \wedge y))))$	SD-$\vee$, SD-$\wedge$
H79	$x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge ((x \wedge (y \vee (x \wedge z))) \vee (z \wedge u))$	SD-$\vee$, SD-$\wedge$
H80	$(x \wedge y) \vee (x \wedge z) = x \wedge ((x \wedge y) \vee (z \wedge (x \vee (y \wedge (x \vee z)))))$	CM-$\vee$
H82	$(x \wedge y) \vee (x \wedge z) = x \wedge ((y \wedge (x \vee z)) \vee (z \wedge (x \vee y)))$	CD-$\vee$, CM-$\vee$

Table 1: Huntington Identities Justified by Huntington Implications

4 More Huntington Identities

This section contains several non-modular Huntington identities that do not satisfy the Huntington implications (SD-$\vee$), (CD-$\vee$), (CM-$\vee$), or their duals.

Theorem 3. *The variety of lattices defined by*

$$x \wedge (y \vee z) = x \wedge (y \vee ((x \vee y) \wedge (z \vee (x \wedge y)))) \tag{H58}$$

is a non-modular Huntington variety.

Proof. (The automatic proof from which this proof was derived is given in the Appendix.) We show that any uniquely complemented lattice satisfying (H58) also satisfies the order reversibility property $a \leq b \Rightarrow b' \leq a'$. Assume $a \leq b$ and therefore $a \wedge b' = 0$. In (H58), set $x = a$, $y = b'$ and $z = (a \vee b')'$, then simplify the right-hand side, giving

$$a \wedge (b' \vee (a \vee b')') = 0.$$

Then unique complementation gives $b' \vee (a \vee b')' = a'$, and therefore $b' \leq a'$. Thus the unary mapping $x \mapsto x'$ is order reversible, and it is well known that this forces distributivity of a uniquely complemented lattice (see [13, p. 48, Cor. 1]; for a computer proof see [12]). □

Along with (H58), additional Huntington identities not satisfying the Huntington implications are shown in the following list. Automated proofs are given on the supporting Web page [12].

$$x \wedge (y \vee (z \wedge (x \vee u))) = x \wedge (y \vee (z \wedge (x \vee (z \wedge u)))) \tag{H1}$$
$$x \wedge (y \vee (x \wedge z)) = x \wedge (y \vee (z \wedge ((x \wedge (y \vee z)) \vee (y \wedge z)))) \tag{H2}$$
$$x \wedge (y \vee (x \wedge z)) = x \wedge (y \vee (z \wedge (y \vee (x \wedge (z \vee (x \wedge y)))))) \tag{H3}$$
$$x \vee (y \wedge (x \vee z)) = x \vee (y \wedge (z \vee (x \wedge (z \vee y)))) \tag{H55}$$
$$x \wedge (y \vee z) = x \wedge (y \vee ((x \vee y) \wedge (z \vee (x \wedge y)))) \tag{H58}$$

5 Covers of N_5

Since we are interested in discovering non-modular lattice identities which force distributivity under unique complementation, we naturally look at all the covers of the variety $\mathcal{V}(N_5)$. Ralph McKenzie [9] constructed the fifteen lattices L1–L15, shown in Figure 1, whose varieties are join-irreducible covers of the least non-modular variety $\mathcal{V}(N_5)$. It is a deep result in lattice theory that there are exactly 16 covers of $\mathcal{V}(N_5)$: the fifteen varieties of McKenzie and the trivial $\mathcal{V}(M_3) \vee \mathcal{V}(N_5)$ (Figure 2). The results in this paper demonstrate that all these sixteen varieties are, in fact, Huntington varieties. Table 2 lists the lattices from Figures 1 and 2 for which the Huntington identities given in the paper hold. For more information on these non-modular lattice laws and other details, see [10].

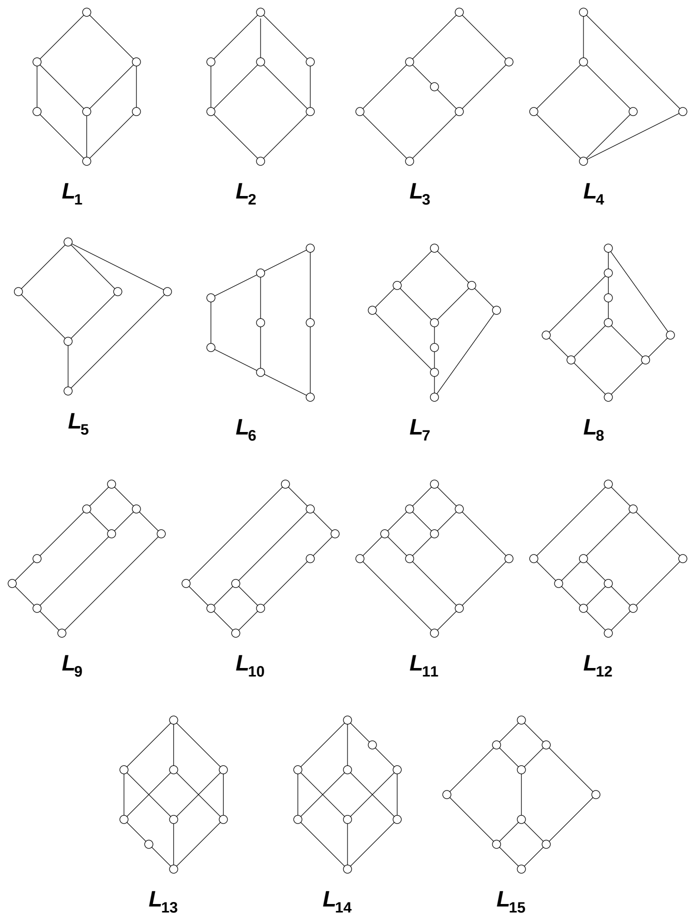

Figure 1: All Covers of the Least Non-Modular Lattice N_5

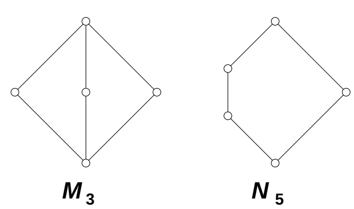

Figure 2: Lattices M_3 and N_5

(H1)	1	2					7			10	11		13	14	15		N_5
(H2)	1			4		6	7	8	9	10	11	12	13	14	15	M_3	N_5
(H3)	1			4		6	7	8	9	10	11	12	13	14	15	M_3	N_5
(H18)	1			4		6	7		9	10	11		13		15	M_3	N_5
(H50)	1					6	7		9	10	11		13	14	15		N_5
(H51)	1						7			10	11		13	14	15		N_5
(H55)		2			5	6	7	8	9	10		12	13	14	15	M_3	N_5
(H58)		2			5	6	7	8	9	10		12	13	14	15	M_3	N_5
(H64)		2				6	7	8	9	10	11	12	13	14	15		N_5
(H68)		2				6	7	8	9	10		12	13	14	15		N_5
(H69)		2				6	7	8	9	10		12	13	14	15		N_5
(H76)						6	7	8	9	10			13	14	15		N_5
(H79)							7			10			13	14	15		N_5
(H80)	1		3	4		6	7	8	9	10	11		13		15	M_3	N_5
(H82)	1			4		6	7		9	10	11		13		15		N_5

Table 2: Lattices ($L_1 - L_{15}$, M_3, N_5) for which the Identities Hold

6 Methods of Discovery

Two types of method were used to find the Huntington identities presented in this paper. The first was to automatically generate a great number of candidates and submit each to an automated theorem proving program, and the second are canonical representation techniques.

The automatically generated candidates were constructed under the following constraints. Each candidate must (1) be a lattice equation in terms of meet and join, (2) not necessarily hold for all lattices, (3) hold for all Boolean algebras, and (4) hold for the least non-modular lattice N_5. Several thousand identities were generated, and about 80 were proved to be Huntington identities. Some of the proofs (e.g., H58 of Theorem 3) were easy for the theorem provers, and some were difficult, requiring some human guidance. The Huntington identities were then classified according to the non-modular lattices in which they hold, as in Table 2. Those that could be proved to be equivalent (mod lattice theory and duality) to others were removed, and in some cases, if implications (mod lattice theory and duality) could be proved, the stronger ones were removed. The result is the set listed in Table 2.

Classical lattice theory is abundant with representation techniques for finding non-modular lattice identities. Most of these originate from the seminal paper of Ralph Mckenzie [8]. For example, he gives equational bases for various covers of $\mathcal{V}(N_5)$. These bases alone lead to several non-modular Huntington laws. For example, (H69) discussed in the beginning of the paper is simply a weaker 3-variable version of McKenzie's four-variable identity (η_2) [9, p.7]. Naturally, this weaker version holds in a several covers of N_5. Another technique is to start from a canonical term, say $x \wedge (y \vee z)$, and consider lattice identities of the form $f = g$ where the lattice terms f and g are members of $FL(N_5)$ having the same homomorphic image $x \wedge (y \vee z)$ under f where $f : FL(N_5, 3) \mapsto FL(D, 3)$

is the canonical homomorphism. Another rich source is that of Jónsson and Rival [5] who gave a family of lattice laws which imply either SD-$\vee$ or SD-$\wedge$. We selected several of those and experimented with theorem-proving programs to obtain more Huntington laws. Note that all these identities are, of course, non-modular.

Appendix

Proof of Theorem 2, Part CD-$\vee$

This proof was produced by the program Prover9 [8]. The input and output files can be found on the supporting Web page [12].

Notes on the Prover9 and Otter proofs.

1. Proofs are by contradiction.
2. Terms x, y, z, u, v, w are variables, and A, B, C, D are constants.
3. Implications are written as disjunctions.
4. The justification "$m \rightarrow n$" means that an instance of clause m is used to replace a term in an instance of clause n, and "$; i, j, \cdots$" means simplification with $i, j, \cdots$. The justification "$n_1 n_2 \cdots$" indicates application of an implication.

13	$x \vee y = y \vee x$	[input]
14	$x \wedge y = y \wedge x$	[input]
15	$(x \vee y) \vee z = x \vee (y \vee z)$	[input]
16	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	[input]
17	$x \wedge (x \vee y) = x$	[input]
18	$x \vee (x \wedge y) = x$	[input]
19	$x \vee x' = 1$	[input]
20	$x \wedge x' = 0$	[input]
21	$x \vee y \neq 1 \;\mid\; x \wedge y \neq 0 \;\mid\; x' = y$	[input]
22	$A \wedge B = A$	[input]
23	$A' \vee B' \neq A'$	[input]
24	$x \vee y \neq x \vee z \;\mid\; x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	[input]
26	$x \wedge (y \wedge z) = y \wedge (x \wedge z)$	[14 $\rightarrow$ 16; 16]
32	$x \vee ((x \wedge y) \vee z) = x \vee z$	[18 $\rightarrow$ 15]
37	$x \vee (x' \vee y) = 1 \vee y$	[19 $\rightarrow$ 15]
39	$x \wedge 1 = x$	[19 $\rightarrow$ 17]
42	$x \vee 0 = x$	[20 $\rightarrow$ 18]
43	$A \wedge (B \wedge x) = A \wedge x$	[22 $\rightarrow$ 16]
51	$x \vee y \neq 1 \;\mid\; x \wedge (y \vee x') = 0 \vee (x \wedge y)$	[19 $\rightarrow$ 24; 20 13]
60	$1 \wedge x = x$	[39 $\rightarrow$ 14]
66	$0 \vee x = x$	[42 $\rightarrow$ 13]
69	$x \vee y \neq 1 \;\mid\; x \wedge (y \vee x') = x \wedge y$	[51; 66]

72	$1 \vee x = 1$	[60 → 17]
73	$x \vee (x' \vee y) = 1$	[37; 72]
75	$0 \wedge x = 0$	[66 → 17]
79	$x \wedge (y \wedge x') = y \wedge 0$	[20 → 26]
81	$x \vee 1 = 1$	[72 → 13]
83	$x \wedge 0 = 0$	[75 → 14]
84	$x \wedge (y \wedge x') = 0$	[79; 83]
103	$A \wedge (A' \vee B') \neq 0$	[21 73 23]
170	$x \vee (x \wedge y)' = 1$	[19 → 32; 81]
185	$x \vee (y \wedge x)' = 1$	[14 → 170]
194	$B \vee A' = 1$	[22 → 185]
891	$B \wedge (A' \vee B') = B \wedge A'$	[69 194]
5852	$A \wedge (A' \vee B') = 0$	[891 → 43; 84]
5853	$\square$	[5852 103]

Proof of Theorem 3

This proof was produced by the program Prover9 [8]. The input and output files can be found on the supporting Web page [12].

29	$x \vee y = y \vee x$	[input]
37	$x \vee (y \wedge x) = x$	[input]
38	$x \vee (x \vee y) = x \vee y$	[input]
40	$x \wedge x' = 0$	[input]
47	$x \vee 0 = x$	[input]
48	$0 \vee x = x$	[input]
49	$x \vee y \neq 1 \ \mid\ x \wedge y \neq 0 \ \mid\ x' = y$	[input]
50	$x \wedge (x \vee y)' = 0$	[input]
51	$x \vee (y \vee (x \vee y)') = 1$	[input]
52	$x \wedge (y \vee ((x \vee y) \wedge (z \vee (x \wedge y)))) = x \wedge (y \vee z)$	[input]
53	$A \wedge B = A$	[input]
54	$A' \vee B' \neq A'$	[input]
103	$A \vee B = B$	[53 → 37; 29]
107	$A \wedge B' = 0$	[103 → 50]
109	$A \wedge (B' \vee ((A \vee B') \wedge x)) = A \wedge (B' \vee x)$	[107 → 52; 47]
2392	$A \wedge (B' \vee (A \vee B')') = 0$	[40 → 109; 29 48 107]
2439	$B' \vee (A \vee B')' = A'$	[49 51 2392]
2481	$A' \vee B' = A'$	[2439 → 38; 29 2439]
2482	$\square$	[2481 54]

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