More First-order Test Problems in Math and Logic

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Problem Summary

- Condensed Detachment Problems
  - xcb-reflex, xcb, yqe, mv-dist-star, twoval-luka-23, twoval-mer-21
  - sheffer-org-d23, mingle-bci, mingle-concise, intuit-imp
- Equational Bases for Boolean Algebra
  - ba-dn1, sheffer-sh1, sheffer-mstar, sheffer-cstar
- Equational Lattice Theory
  - ol-e51, ol-e62, ol-rw1, ol-rw2, oml-mod, lattice-uc
- Miscellaneous Equality Problems
  - cs-comm-ad, cs-comm-dn, cs-comm-na, hbck

xcb-reflex [ec-nd:1983,bf-to-lw:1998]

- Prove reflexivity from formula XCB by condensed detachment. This was long thought to be a nontheorem and was first proved by Fitelson.
- Status: theorem (unsatisfiable).

```
-P(e(x,y)) | -P(x) | P(y).          % condensed detachment
P(e(x,e(e(e(x,y),e(z,y)),z))).     % XCB
-P(e(A,A)).                        % denial of reflexivity
```

xcb [ec-nd:1983,lw-xcb:2002]

- Show that formula XCB is a single axiom for the equivalential calculus by deriving the well known single axiom WN by condensed detachment. This was the last remaining open candidate of length 11. It was first proved to be a single axiom by Wos in April 2002.

- Status: theorem (unsatisfiable).

```
-P(e(x,y)) | -P(x) | P(y).          % condensed detachment
P(e(x,e(e(e(x,y),e(z,y)),z))).     % XCB

-P(e(e(A,e(B,C)),e(C,e(A,B)))).    % denial of single axiom WN
```

yqe [wm-lg:1992,tu-to-lw:2001]

- Is formula YQE a single axiom for the right group calculus? Any counter-models must be infinite.
- Status: open.

```
-P(e(x,y)) | -P(x) | P(y).                    % condensed detachment
P(e(e(x,y),e(e(x,z),e(y,z)))).                % YQE

-P(e(A,e(A,e(e(B,C),e(e(B,D),e(C,D)))))). % denial of single axiom L2'
```

mv-dist-star [bf-kh-dist:2001,bv-dist:2001]

- From the axioms MV1–MV4 (many-valued sentential calculus), derive the distributivity properties AK1, AK2, KA1, and KA2. This is four separate problems.

- Status: all are theorems (unsatisfiable).

```
-P(i(x,y)) | -P(x) | P(y).        % condensed detachment

P(i(x,i(y,x))).                   % MV1
P(i(i(x,y),i(i(y,z),i(x,z)))).    % MV2
P(i(i(i(x,y),y),i(i(y,x),x))).    % MV3
P(i(i(n(x),n(y)),i(y,x))).        % MV4

-P(i(i(i(A,n(i(i(n(B),n(C)),n(C)))),n(i(i(n(B),n(C)),n(C)))),n(i(i(n(
   i(i(A,B),B)),n(i(i(A,C),C))),n(i(i(A,C),C)))))).  % AK1
-P(i(n(i(i(n(i(i(A,B),B)),n(i(i(A,C),C))),n(i(i(A,C),C)))),i(i(A,n(i(
   i(n(B),n(C)),n(C)))),n(i(i(n(B),n(C)),n(C)))))).  % AK2
-P(i(n(i(i(n(A),n(i(i(B,C),C))),n(i(i(B,C),C)))),i(i(n(i(i(n(A),n(B)),
   n(B))),n(i(i(n(A),n(C)),n(C)))),n(i(i(n(A),n(C)),n(C)))))). % KA1
-P(i(i(i(n(i(i(n(A),n(B)),n(B))),n(i(i(n(A),n(C)),n(C)))),n(i(i(n(A),
   n(C)),n(C)))),n(i(i(n(A),n(i(i(B,C),C))),n(i(i(B,C),C)))))). % KA2
```

twoval-luka-23 [lw-bf:2001]

- Show that the formula Luka-23 is a single axiom for two-valued logic by deriving the Łukasiewicz 3-basis.
- Status: theorem (unsatisfiable).

```
-P(i(x,y)) | -P(x) | P(y).            % condensed detachment
P(i(i(i(x,y),i(i(i(n(z),n(u)),v),z)),i(w,i(i(z,x),i(u,x))))). % Luka-23

% denial of Lukasiewicz 3-basis

-P(i(i(A,B),i(i(B,C),i(A,C)))) |
          -P(i(i(n(A),A),A)) |
          -P(i(A,i(n(A),B))).
```

twoval-mer-21 [lw-msax:2001]

- Show that the formula Mer-21 is a single axiom for two-valued logic by deriving the Łukasiewicz 3-basis.
- Status: theorem (unsatisfiable).

```
-P(i(x,y)) | -P(x) | P(y).              % condensed detachment
P(i(i(i(i(i(x,y),i(n(z),n(u))),z),v),i(i(v,x),i(u,x)))). % Mer-21

% denial of Lukasiewicz 3-basis

-P(i(i(A,B),i(i(B,C),i(A,C)))) |
          -P(i(i(n(A),A),A)) |
          -P(i(A,i(n(A),B))).
```

sheffer-org-d23 [bf-kh-sheffer:2001]

- Show that formula ORG-D23 is a single axiom for propositional calculus in terms of the Sheffer stroke by deriving Nicod's single axiom. The nice property of ORG-D23 is that it is organic; that is, no subformula is a theorem.

- Status: theorem (unsatisfiable).

```
-P(d(x,d(y,z))) | -P(x) | P(z).  % detachment for Sheffer stroke
P(d(d(x,d(y,z)),d(d(x,d(y,z)),d(d(u,z),d(d(z,u),d(x,u)))))). % ORG-D23

% denial of Nicod's original single axiom

-P(d(d(A,d(B,C)),d(d(E,d(E,E)),d(d(F,B),d(d(A,F),d(A,F)))))).
```

mingle-bci [ze-tv:2001]

- Show that if the mingle formula is added to the logic BCI, the Karpenko formula can be derived by condensed detachment.
- Status: theorem (unsatisfiable).

```
-P(i(x,y)) | -P(x) | P(y).            % condensed detachment
P(i(i(x,y),i(i(z,x),i(z,y)))).       % B
P(i(i(x,i(y,z)),i(y,i(x,z)))).       % C
P(i(i(i(i(i(x,y),y),x),z),i(i(i(i(i(y,x),x),y),z),z))).  % mingle

% denial of Karpenko formula

-P(i(i(A,i(i(B,B),A)),i(i(i(A,B),B),i(i(B,A),A)))).
```

mingle-concise [efhw:2001]

- Show that the mingle axiom can be derived from the three formulas given below by condensed detachment. This gives a simpler basis for the system RM$\rightarrow$.
- Status: theorem (unsatisfiable).

```
-P(i(x,y)) | -P(x) | P(y).              % condensed detachment
P(i(i(x,y),i(i(y,z),i(x,z)))).         % suffixing
P(i(x,i(i(x,y),y))).                   % assertion
P(i(i(i(i(i(x,y),z),i(y,x)),z),z)).    % candidate

% denial of mingle axiom

-P(i(i(i(i(i(A,B),B),A),C),i(i(i(i(i(B,A),A),B),C),C))).
```

intuit-imp [tu-to-bf:2001]

- Show that the candidate formula is not a single axiom for intuitionistic implication by finding a model in which the required property below fails.
- Status: nontheorem (satisfiable).

```
-P(i(x,y)) | -P(x) | P(y).                    % condensed detachment
P(i(i(x,y),i(i(y,i(i(z,x),u)),i(x,u)))).     % candidate

-P(i(A,i(B,A))).                             % required property
```

ba-dn1 [basax:2000]

- Show that equation DN-1 is a single axiom for Boolean algebra in terms of disjunction and negation by deriving the Huntington 3-basis.
- Status: theorem (unsatisfiable).

```
n(n(n(x + y) + z) + n(x + n(n(z) + n(z + u)))) = z.     % DN-1

% denial of Huntington 3-basis

B + A != A + B |
          (A + B) + C != A + (B + C) |
          n(n(A) + B) + n(n(A) + n(B)) != A.
```

sheffer-sh1 [basax:2000,sw:2000]

- Show that equation Sh-1 is a single axiom for Boolean algebra in terms of the Sheffer stroke by deriving the Meredith 2-basis.
- Status: theorem (unsatisfiable).

```
f(f(x,f(f(y,x),x)),f(y,f(z,x))) = y.  % Sh-1

% denial of Meredith 2-basis

f(f(A,A),f(B,A)) != A | f(A,f(B,f(A,C))) != f(f(f(C,B),B),A).
```

sheffer-mstar [basax:2000]

- Show that each of the equations below is too weak to be a single axiom for Boolean algebra in terms of the Sheffer stroke. This is 7 separate problems.

- Status: all are nontheorems (satisfiable).

```
f(f(y,f(f(x,z),y)),f(x,f(z,y))) = x.    % M5A
f(f(f(y,f(x,z)),y),f(x,f(z,y))) = x.    % M5B
f(f(f(y,f(x,y)),y),f(x,f(y,z))) = x.    % M6A
f(f(y,f(y,f(x,y))),f(x,f(y,z))) = x.    % M6B
f(f(f(y,f(x,x)),y),f(x,f(y,z))) = x.    % M6C
f(f(y,f(y,f(x,x))),f(x,f(y,z))) = x.    % M6D
f(f(f(y,f(z,x)),y),f(x,f(z,y))) = x.    % M8A

% denial of Meredith 2-basis

f(f(A,A),f(B,A)) != A | f(A,f(B,f(A,C))) != f(f(f(C,B),B),A).
```

sheffer-cstar [basax:2000,sw:2000]

- Which of equations C1–C16 are single axioms for Boolean algebra in terms of the Sheffer stroke? There are 16 problems here: take each of the candidates and pair it with the denial.

- Status: all are open.

```
f(f(y,f(f(x,y),y)),f(x,f(y,z))) = x.   % C1
f(f(y,f(y,f(x,y))),f(x,f(z,y))) = x.   % C2
f(f(y,f(y,f(y,x))),f(x,f(z,y))) = x.   % C3
f(f(y,f(y,f(y,x))),f(x,f(y,z))) = x.   % C4
f(f(y,f(y,f(x,z))),f(x,f(z,y))) = x.   % C5
f(f(y,f(y,f(z,x))),f(x,f(y,z))) = x.   % C6
f(f(y,f(y,f(x,x))),f(x,f(z,y))) = x.   % C7
f(f(f(y,f(y,x)),y),f(x,f(z,y))) = x.   % C8
f(f(f(y,f(x,x)),y),f(x,f(z,y))) = x.   % C9
f(f(f(y,f(x,z)),y),f(x,f(y,z))) = x.   % C10
f(f(f(y,f(z,x)),y),f(x,f(y,z))) = x.   % C11
f(f(f(y,f(y,x)),y),f(x,f(y,z))) = x.   % C12
f(f(f(f(y,x),y),y),f(x,f(z,y))) = x.   % C13
```

```
f(f(f(f(y,x),y),y),f(x,f(y,z))) = x.   % C14
f(f(f(f(y,x),z),z),f(x,f(y,z))) = x.   % C15
f(f(f(f(y,x),z),z),f(x,f(z,y))) = x.   % C16

% denial of Meredith 2-basis

f(f(A,A),f(B,A)) != A | f(A,f(B,f(A,C))) != f(f(f(C,B),B),A).
```

ol-e51 [wm-to-nm:1999]

- Show that equation E51 does not necessarily hold in ortholattices.
- Status: nontheorem (satisfiable).

```
x ^ y = y ^ x.                  % lattice axioms
(x ^ y) ^ z = x ^ (y ^ z).
x ^ (x v y) = x.
x v y = y v x.
(x v y) v z = x v (y v z).
x v (x ^ y) = x.

c(x) ^ x = Zero.                % add these for ortholattice
c(x) v x = One.
x ^ y = c(c(x) v c(y)).

% denial of E51
((A v c(B))^ (((A^B) v (c(A)^B)) v (c(A)^c(B)))) !=
       ((A^B) v (c(A)^c(B))).
```

ol-e62 [wm-to-nm:1999]

- Show that equation E62 does not necessarily hold in ortholattices.
- Status: nontheorem (satisfiable).

```
x ^ y = y ^ x.                  % lattice axioms
(x ^ y) ^ z = x ^ (y ^ z).
x ^ (x v y) = x.
x v y = y v x.
(x v y) v z = x v (y v z).
x v (x ^ y) = x.

c(x) ^ x = Zero.                % add these for ortholattice
c(x) v x = One.
x ^ y = c(c(x) v c(y)).

% denial of E62
A ^ (B v (A ^ (c(A) v (A ^ B)))) != A ^ (c(A) v (A ^ B)).
```

ol-rw1 [mr-kw:2001]

- Show that equation *3-68 does not necessarily hold in weak orthomodular lattices.
- Status: nontheorem (satisfiable).

```
x ^ y = y ^ x.                  % lattice axioms
(x ^ y) ^ z = x ^ (y ^ z).
x ^ (x v y) = x.
x v y = y v x.
(x v y) v z = x v (y v z).
x v (x ^ y) = x.

c(x) ^ x = Zero.                % add these for ortholattice
c(x) v x = One.
x ^ y = c(c(x) v c(y)).

(c(x) ^ (x v y)) v (c(y) v (x ^ y)) = One.  % weak orthomodular law

A ^ (B v (A ^ (c(A) v (A ^ B)))) != A ^ (c(A) v (A ^ B)).  % *3-68
```

ol-rw2 [mr-kw:2001]

- Show that equation 98A does not necessarily hold in ortholattices.
- Status: nontheorem (satisfiable).

```
x ^ y = y ^ x.                  % lattice axioms
(x ^ y) ^ z = x ^ (y ^ z).
x ^ (x v y) = x.
x v y = y v x.
(x v y) v z = x v (y v z).
x v (x ^ y) = x.

c(x) ^ x = Zero.                % add these for ortholattice
c(x) v x = One.
x ^ y = c(c(x) v c(y)).

% denial of equation 98A.

A v (c(B) ^ (c(A) v (c(B) ^ (A v (c(B) ^ c(A))))))  !=
A v (c(B) ^ (c(A) v (c(B) ^ (A v (c(B) ^ (c(A) v (c(B) ^ A))))))).
```

oml-mod [mr-kw:2001]

- Show that orthomodular lattices are not necessarily modular. This is well known, but it is a good test problem for finite model search.
- Status: nontheorem (satisfiable).

```
x ^ y = y ^ x.                  % lattice axioms
(x ^ y) ^ z = x ^ (y ^ z).
x ^ (x v y) = x.
x v y = y v x.
(x v y) v z = x v (y v z).
x v (x ^ y) = x.

c(x) ^ x = Zero.                % add these for ortholattice
c(x) v x = One.
x ^ y = c(c(x) v c(y)).

x v (c(x) ^ (x v y)) = x v y.  % orthomodular law (OM)

A v (B ^ (A v C)) != (A v B) ^ (A v C).  % denial of modularity
```

lattice-uc [mpv-ws:2002]

- Consider uniquely complemented (UC) lattices. We are looking for weak properties that force them to be Boolean. Distributivity is well known to do the job, so we use it as our goal (denial). There are six problems here. For the open ones, all counterexamples are infinite, because all finite uniquely complemented lattices are Boolean.

- Status: three are theorems, three are open.

```
x ^ y = y ^ x.                                  % lattice axioms
(x ^ y) ^ z = x ^ (y ^ z).
x ^ (x v y) = x.
x v y = y v x.
(x v y) v z = x v (y v z).
x v (x ^ y) = x.

x v c(x) = One.                                 % complementation
x ^ c(x) = Zero.
```

```
x v y != One | x ^ y != Zero | c(x) = y.  % complements are unique

A ^ (B v C) != (A ^ B) v (A ^ C).         % denial of distributivity

% Take each of the following 6 with the preceding clauses.

% Each of these three gives a theorem.

x^(y v (z^(x v (y^z)))) = x^(y v (x^z)).              % 94-6
x^((y^(z v (x^y))) v (z^(x v y))) = (x^y) v (x^z).    % 94-37
x^(y v (z v (u^(x v (y^z))))) = x^(y v (z v (x^u))).  % G61

% Each of these three is open.
x^(y v ((x v y)^(z v (y^(x v z))))) = x^(y v z).      % 94-3
x^(y v (z^(u v (x^(y v z))))) = x^(y v (z^(x v u))).  % F53
x^(y v (z^((x^z) v (u^(y v z))))) = x^(y v ((x^z) v (z^u))). % G113
```

cs-comm-ad [kurosh:1955,mpv-ws:2002]

- Assume a cancellative semigroup (CS) admits a commutator operation. Then the following three properties are equivalent: (1) commutator is associative; (2) commutator distributes over product; (3) the semigroup is nilpotent of class 2. This is a generalization of the corresponding theorem for group theory. The problem here is to prove (1) implies (2).

- Status: theorem (unsatisfiable).

```
(x * y) * z = x * (y * z).          % product is associative
x * y != x * z | y = z.             % left cancellation
y * x != z * x | y = z.             % right cancellation

x * y = y * (x * (x @ y)).          % CS admits commutator
(x @ y) @ z = x @ (y @ z).          % commutator is associative

(a * b) @ c != (a @ c) * (b @ c). % denial: comm. distributes over prod
```

cs-comm-dn [kurosh:1955,mpv-ws:2002]

- See the description of problem cs-comm-ad. The problem here is to prove that the distributivity property (2) implies the nilpotent property (3).
- Status: theorem (unsatisfiable).

```
(x * y) * z = x * (y * z).          % product is associative
x * y != x * z | y = z.             % left cancellation
y * x != z * x | y = z.             % right cancellation

x * y = y * (x * (x @ y)).          % CS admits commutator
(x * y) @ z = (x @ z) * (y @ z).    % commutator distributes over product

(a @ b) * c != c * (a @ b).         % denial: nilpotent class 2
```

cs-comm-na [kurosh:1955,mpv-ws:2002]

- See the description of problem cs-comm-ad. The problem here is to prove that the nilpotent property (3) implies the associativity property (1).

- Status: theorem (unsatisfiable).

```
(x * y) * z = x * (y * z).          % product is associative
x * y != x * z | y = z.             % left cancellation
y * x != z * x | y = z.             % right cancellation

x * y = y * (x * (x @ y)).          % CS admits commutator
(x @ y) * z = z * (x @ y).          % nilpotent class 2

(a @ b) @ c != a @ (b @ c).         % denial: commutator is associative
```

hbck [wm-rp:1996,bv-hbck:2002]

- Axioms for the quasivariety HBCK are given below. Show that equation J follows. This result has been known for some time by a model-theoretic argument. The first first-order proof was found by Veroff in 2002.

- Status: theorem (unsatisfiable).

```
x * One = One.                                  % M3
One * x = x.                                    % M4
(x * y) * ((z * x) * (z * y)) = One.            % M5
x * y != One | y * x != One | x = y.            % M7
x * x = One.                                    % M8
x * (y * z) = y * (x * z).                      % M9
(x * y) * (x * z) = (y * x) * (y * z).          % H

(((A * B) * B) * A) * A != (((B * A) * A) * B) * B.  % denial of J
```