University of New Mexico
Department of Computer Science

Final Examination

CS 261 Mathematical Foundations of Computer Science
Spring, 2010

Name:
Email:

- *"Nothing is true. All is permitted"* - Friedrich Nietzsche. Well, not exactly. **You are not permitted to discuss this exam with any other person.** If you do so, you will surely be smitten (collusion on any problem will result in a 0 on the test). However, you may consult any *other* sources including books, papers, web pages, computational devices, animal entrails, seraphim, cherubim, etc. in your quest for truth and solutions. Please acknowledge your sources.
- *Show your work!* You will not get full credit if we cannot figure out how you arrived at your answer. A numerical solution obtained via a computer program is unlikely to get much credit without a correct mathematical derivation.
- Write your solution in the space provided for the corresponding problem. Do NOT use any extra paper for your final solutions.
- If any question is unclear, ask for clarification.

Question	Points	Score	Grader
1	10		
2	20		
3	20		
4	20		
5	10		
6	20		
Total	100		

1. **Logic**

   The amazing Criswell has made the following three predictions about the next academic year:

   - A: The wood-elves will rise again!
   - B: Air Estonia will partner with Irkutsk International in an exciting new frequent flyer program!
   - It will not be the case that my first prediction occurs but not my second!

   Let A be the first proposition and B be the second proposition.

   (a) (3 points) Write the third proposition as a logical combination of A and B

   (b) (7 points) Prove, using propositional logic, that at least one of the Amazing Criswell's predictions will hold.

2. **Graphs**

A valid edge coloring of a graph $G = (V, E)$ assigns a color to each edge $(u, v) \in E$ that is different than any other edge (u, v) touches. Specifically, for all edges (u, v) (u, v) must have a different color from any edge $(u, w) \in E$ and from any edge $(x, v) \in E$ for vertices $x, w \in V$. In the edge coloring problem, you want to find a valid edge coloring that uses the minimum number of colors.

For this problem, you will describe a simple recursive algorithm to edge-color a graph with $O(\Delta)$ colors. (Note: Vizing's algorithm shows this can be done with $\Delta + 1$ colors but the algorithm is quite complicated. In this problem, we are looking for a much shorter and simpler algorithm.)

(a) (4 points) Imagine you are faced with the following problem. You have n machines in a network. Some set of pairs of these machines want to exchange files. However, in every time step, each machine can be engaged in just one file exchange. The goal of the problem is to do all the file exchanges as quickly as possible. How would you formulate this as an edge coloring problem? What are the nodes, what are the edges and what do the colors represent?

(b) (10 points) Give a simple recursive algorithm for edge coloring a graph with $O(\Delta)$ colors.

(c) (6 points) What is the maximum number of colors your algorithm uses as a function of Δ. Give explicit constants, not big-O notation for this. What is the runtime of your algorithm on a graph containing n nodes and m edges with max degree Δ, give in big-O notation as a function of n, m and Δ?

3. **Viruses**

A certain computer virus spreads as follows. In a given minute, the number of machines infected at the last minute each infect one new machine. However, the number of machines infected 2 minutes ago are all disinfected. In this problem, you will write and solve a recurrence relation for the number of machines infected after n minutes.

(a) (5 points) Let $f(n)$ be the number of machines that are infected at minute n. Write a recurrence relation for this value.

(b) (5 points) Now find the general form solution to this recurrence using annihilators.

(c) (5 points) Now assume $f(0) = 1$ and $f(1) = 2$. What is the exact solution for $f(n)$?

(d) (5 points) Consider the following second computer virus. In a given minute, only *half* of the number of machines infected at the last minute each infect one new machine. However, no machines are ever disinfected. What is the general form of the solution for this second virus? Which virus spreads more quickly, the first or the second?

4. **Match**

The game of "Match" is played with a special deck of 27 cards. Each card has three attributes: color, shape and number. The possible color values are red, blue or green; the possible shape values are square, circle, heart; and the possible number values are 1, 2 or 3. All of the $3 * 3 * 3 = 27$ possible combinations is represented as a card in the deck. In this game, a *match* is a set of 3 cards with the property that for every one of the three attributes, either all the cards have the same value for that attribute or they all have different values for that attribute. For example, the following three cards are a match (1,red,square), (2, blue, square), (3, green,square).

(a) (8 points) What is the probability that three cards turned over from a well-shuffled deck will form a match? Justify your answer. Hint: Imagine that two cards have already been turned over. How many cards remaining in the deck could form a match with these two cards? (It's the same number no matter what the first two cards are)

(b) (8 points) If n cards are flipped over from a well-shuffled deck, what is the expected number of matches that can be found? Show your work!

(c) (4 points) How many cards do you need to flip over before you would expect to find at least one match?

5. **Auctions**

(10 points) Imagine you are working for a company that puts on the following type of auction. There are n players and each has a unique bid for an item. The bids are considered in an order selected uniformly at random. When each bid is considered, if it is greater than all previous bids considered, a special bell is rung. For example, if there are four bids, 1, 2, 3, 4, and they are considered in the following order: 3, 2, 1, 4, then the bell is rung twice. What is the expected number of times the bell is rung as a function of n? (Your company wants to know this since they are considering giving out prizes whenever the bell is rung.) Hint: Linearity of Expectation.

6. **Primes and Probability**

(a) (2 points) Show that for any integer x, x factors into at most $\log x$ primes. Hint: 2 is the smallest prime.

(b) (5 points) Let x be some positive integer and let p be a prime chosen uniformly at random from all primes less than or equal to m. What is an upper bound on the probability that $p|x$? You can use asymptotic notation. Hint: Use the prime number theorem to determine the number of possible choices for p.

(c) (5 points) Now let x and y both be numbers less than n and let p be a prime chosen uniformly at random from all primes less than or equal to m. Using the previous result, give the probability that $x \equiv y \pmod{p}$

(d) (3 points) If $m = \log^2 n$ in the previous problem, then what is the probability that $x \equiv y \pmod{p}$. Hint: If you're on the right track, you should be able to show that this probability is "small" i.e. something that goes to 0 as n gets large.

(e) (5 points) Finally, show how to apply this result to the following problem. Alice and Bob both have databases x and y of size n, for n a very large number (think terabytes). They want to check to see if their databases are consistent (i.e. they want to check if they are the same) but Alice does not want to have to send her entire database to Bob. What is an algorithm Alice and Bob can use to check consistency with reasonably good probability by sending a lot fewer bits? How many bits does Alice need to send to Bob as a function of n, and what is the probability of failure, where failure means that this algorithm says the databases are the same but in fact they are different?