1. A graph consists of a set of vertices, $V$, and a set of edges $E$, where each edge in $E$ is a tuple of 2 vertices in $V$. For example, we could have $V = \{a, b, c, d\}$ and $E = \{(a, b), (c, d), (a, d), (a, c)\}$. The degree of a vertex $v \in V$ is the number of edges containing $v$. In the example graph, the degree of $a$ is 3, and the maximum degree of the graph is 3.

A coloring of a graph is an assignment of colors to all vertices in $V$ such that for all $(u, v) \in E$, vertex $u$ has a different color than vertex $v$. A coloring of the example graph with 3 colors is: vertex $a$ gets color 1, vertex $b$ gets color 2, vertex $c$ gets color 2, and vertex $d$ gets color 3. Prove the following.

- Any graph with maximum degree of 3 can be colored with 4 colors. Prove this by induction over $n = |V|$. For your IH, assume that any graph with maximum degree of 3 and less than $n$ vertices can be colored with 4 colors.

2. We say that a graph $G = (V, E)$ is connected if for every pair of vertices $u, v \in V$, there exists a path from $u$ to $v$ using only edges in $G$. To illustrate, the example graph in Problem 1 is connected, but becomes
disconnected if we remove edge \((a, b)\). For a graph \(G = (V, E)\) and vertex \(v \in V\), we define \(G - v\) to be the new graph \((V - v, E')\) where \(E'\) is \(E\) minus all edges containing \(v\). Prove the following:

- Every connected graph \(G = (V, E)\) with \(|V| \geq 2\) has at least two vertices \(x_1\) and \(x_2\), such that \(G - x_i\) is connected for \(i = 1, 2\). Prove this by induction on \(n = |V|\). For your IH, assume that any graph with \(2 < |V| < n\) has at least two vertices \(x_1\) and \(x_2\), such that \(G - x_i\) is connected for \(i = 1, 2\).

3. Prove by induction that \(\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}\) for every \(n > 0\), and all \(r \neq 1\). For your IH, assume that \(\sum_{i=0}^{n-1} r^i = \frac{r^n - 1}{r - 1}\).

4. Consider the recurrence \(f(n) = 3f(n/2) + \sqrt{n}\), where \(f(c) = \Theta(1)\), for constants \(c\). Use the Master method to solve this recurrence.

5. Consider the following function:

\[
\text{int f (int n)}\{
\text{    if (n==0) return 2;}
\text{    else if (n==1) return 5;}
\text{    else}{
\text{        int val = 2*f (n-1);}
\text{        val = val - f (n-2);}
\text{        return val;}
\text{    }}
\text{}\}
\]

(a) Write a recurrence relation for the value returned by \(f\). Solve the recurrence exactly using annihilators. (Don’t forget to check it)

(b) Write a recurrence relation for the running time of \(f\). Get a tight upperbound (i.e. big-O) on the solution to this recurrence using annihilators.