CS 361, Lecture 22

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Outline

- Red Black Trees
- Other Balanced Trees

HW Questions

- Are there any questions on the current HW?

Red-Black Properties

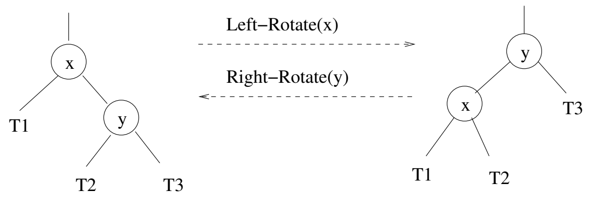
A BST is a red-black tree if it satisfies the RB-Properties

1. Every node is either red or black
2. The root is black
3. Every leaf (NIL) is black
4. If a node is red, then both its children are black
5. For each node, all paths from the node to descendant leaves contain the same number of black nodes

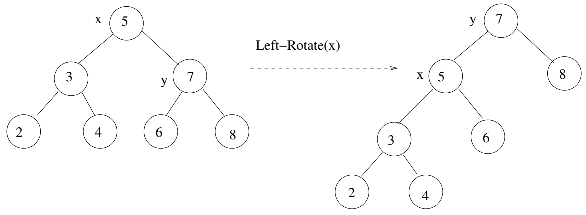
Left-Rotate

- Left-Rotate(x) takes a node x and "rotates" x with its right child
- Right-Rotate is the symmetric operation
- *Both Left-Rotate and Right-Rotate preserve the BST Property*
- We'll use Left-Rotate and Right-Rotate in the RB-Insert procedure

Picture



Example



Binary Search Tree Property

- Let x be a node in a binary search tree. If y is a node in the left subtree of x, then key(y)$\leq$key(x). If y is a node in the right subtree of x then key(y)$\geq$key(x)

In-Class Exercise

Show that Left-Rotate(x) maintains the BST Property. In other words, show that if the BST Property was true for the tree before the Left-Rotate(x) operation, then it's true for the tree after the operation.

- Show that after rotation, the BST property holds for the entire subtree rooted at x
- Show that after rotation, the BST property holds for the subtree rooted at y
- Now argue that after rotation, the BST property holds for the entire tree

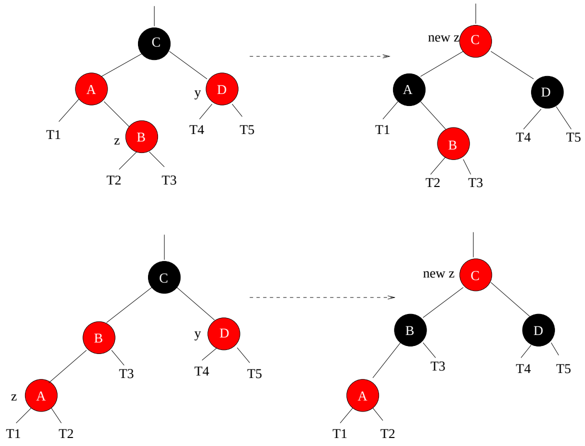
RB-Insert(T,z)

1. Set left(z) and right(z) to be NIL
2. Let y be the last node processed during a search for z in T
3. Insert z as the appropriate child of y (left child if key(z)$\leq$ y, right child otherwise)
4. Color z red
5. Call the procedure RB-Insert-Fixup

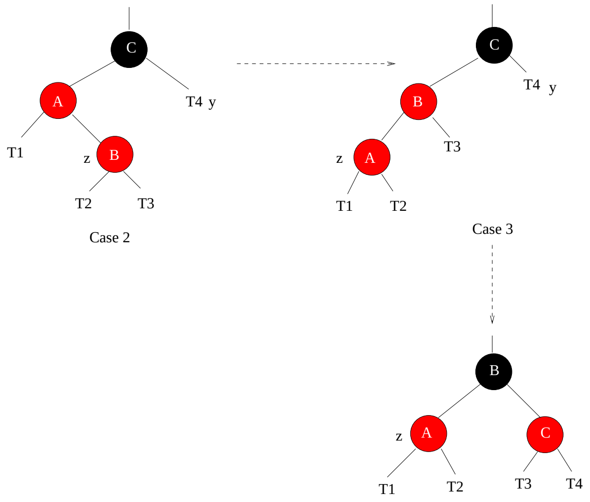
RB-Insert-Fixup(T,z)

```
RB-Insert-Fixup(T,z){
  while (color(p(z)) is red){
    case 1: z's uncle, y, is red{
      do case 1
    }
    case 2: z's uncle, y, is black and z is a right child{
      do case 2
    }
    case 3: z's uncle, y, is black and z is a left child{
      do case 3
    }
  }
  color(root(T)) = black;
}
```

Case 1



Case 2 and 3



Loop Invariant

At the start of each iteration of the loop:

- Node z is red
- If parent(z) is the root, then parent(z) is black
- If there is a violation of the red-black properties, there is at most one violation, and it is either property 2 or 4. If there is a violation of property 2, it occurs because z is the root and is red. If there is a violation of property 4, it occurs because both z and parent(z) are red.

Pseudocode

- Detailed Pseudocode for RB-Insert and RB-Insert-Fixup is in the book, Chapter 13.3
- A detailed proof of correctness for RB-Insert-Fixup in the the same Chapter
- Code for *RB-Deletion* is also in Chapter 13

Other Balanced BSTs

- We'll now *briefly* discuss some other balanced BSTs
- They all implement Insert, Delete, Lookup, Successor, Predecessor, Maximum and Minimum efficiently

AVL Trees

- An AVL tree is height-balanced: For each node x, the heights of the left and right subtrees of x differ by at most 1
- Each node has an additional height field $h(x)$
- Claim: An AVL tree with n nodes has height $O(\log n)$

AVL Trees

- Claim: An AVL tree with n nodes has height $O(\log n)$
- Q: For an AVL tree of height h, how many nodes must it have in it?
- A: We can write a recurrence relation. Let $T(h)$ be the minimum number of nodes in a tree of height h
- Then $T(h) = T(h-1) + T(h-2) + 1$, $T(2) = T(1) \geq 1$
- This is similar to the recurrence relation for Fibonnaci numbers! Solution:

$$T(h) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^h - 2$$

AVL Trees

- So we have the equation $n > T(h)$. Let $\phi = \frac{1+\sqrt{5}}{2}$. Then:

$$\begin{aligned}
n &\geq \frac{1}{\sqrt{5}}(\phi^h) - 2 && (1)\\
\log n &\geq \log(\frac{1}{\sqrt{5}}) + h \log \phi - 1 && (2)\\
\log n - \log(\frac{1}{\sqrt{5}}) + 1 &\geq h \log \phi && (3)\\
C * \log n &\geq h && (4)
\end{aligned}$$

- Where the final inequality holds for appropriate constant C, and for n large enough. The final inequality implies that $h = O(\log n)$

AVL Tree Insertion

- After insert into an AVL tree, the tree may no longer be height-balanced
- Need to "fix-up" the subtrees so that they become height-balanced again
- Can do this using rotations (similar to case for RB-Trees)
- Similar story for deletions

B-Trees

- B-Trees are balanced search trees designed to work well on disks
- B-Trees are *not* binary trees: each node can have many children
- Each node of a B-Tree contains *several* keys, not just one
- When doing searches, we decide which child link to follow by finding the correct interval of our search key in the key set of the current node.

Disk Accesses

- Consider any search tree
- The number of disk accesses per search will dominate the run time
- Unless the entire tree is in memory, there will usually be a disk access every time an arbitrary node is examined
- The number of disk accesses for most operations on a B-tree is proportional to the height of the B-tree
- I.e. The info on each node of a B-tree can be stored in main memory

B-Tree Properties

The following is true for every node x

- x stores keys, $key_1(x), \dots key_l(x)$ in sorted order (nondecreasing)
- x contains pointers, $c_1(x), \dots, c_{l+1}(x)$ to its children
- Let k_i be any key stored in the subtree rooted at the i-th child of x, then $k_1 \le key_1(x) \le k_2 \le key_2(x) \cdots \le key_l(x) \le k_{l+1}$

B-Tree Properties

- All leaves have the same depth
- Lower and upper bounds on the number of keys a node can contain, given as a function of a fixed integer t:
  - Every node other than the root must have $\ge (t-1)$ keys, and t children. If the tree is non-empty, the root must have at least one key (and 2 children)
  - Every node can contain at most $2t-1$ keys, so any internal node can have at most $2t$ children

Note

- The above properties imply that the height of a B-tree is no more than $\log_t \frac{n+1}{2}$, for $t \geq 2$, where n is the number of keys.
- If we make t, larger, we can save a larger (constant) fraction over RB-trees in the number of nodes examined
- A (2-3-4)-tree is just a B-tree with $t = 2$

In-Class Exercise

We will now show that for any B-Tree with height h and n keys, $h \leq \log_t \frac{n+1}{2}$, where $t \geq 2$.

Consider a B-Tree of height $h > 1$

- Q1: What is the minimum number of nodes at depth 1, 2, and 3
- Q2: What is the minimum number of nodes at depth i?
- Q3: Now give a lowerbound for the total number of *keys* (e.g. $n \geq ???$)
- Q4: Show how to solve for h in this inequality to get an upperbound on h

Splay Trees

- A Splay Tree is a kind of BST where the standard operations run in $O(\log n)$ *amortized* time
- This means that over l operations (e.g. Insert, Lookup, Delete, etc), where l is sufficiently large, the total cost is $O(l * \log n)$
- In other words, the average cost per operation is $O(\log n)$
- However a single operation could still take $O(n)$ time
- In practice, they are very fast

Skip Lists

- Technically, not a BST, but they implement all of the same operations
- Very elegant randomized data structure, simple to code but analysis is subtle
- They guarantee that, with high probability, all the major operations take $O(\log n)$ time
- We'll discuss them more next class

High Level Analysis

Comparison of various BSTs

- RB-Trees: + guarantee $O(\log n)$ time for each operation, easy to augment, – high constants
- AVL-Trees: + guarantee $O(\log n)$ time for each operation, – high constants
- B-Trees: + works well for trees that won't fit in memory, – inserts and deletes are more complicated
- Splay Tress: + small constants, – amortized guarantees only
- Skip Lists: + easy to implement, – runtime guarantees are probabilistic only

Which Data Structure to use?

- Splay trees work very well in practice, the "hidden constants" are small
- Unfortunately, they can not guarantee that *every* operation takes $O(\log n)$
- When this guarantee is required, B-Trees are best when the entire tree will not be stored in memory
- If the entire tree will be stored in memory, RB-Trees, AVL-Trees, and Skip Lists are good

Skip List

- Technically, not a BST, but they implement all of the same operations
- Very elegant randomized data structure, simple to code but analysis is subtle
- They guarantee that, with high probability, all the major operations take $O(\log n)$ time

Skip List

- A skip list is basically a collection of doubly-linked lists, $L_1, L_2, \ldots, L_x$, for some integer x
- Each list has a special head and tail node, the keys of these nodes are assumed to be –MAXINT and +MAXINT respectively
- The keys in each list are in sorted order (non-decreasing)

Skip List

- Every key is in the list L_1.
- For all $i > 2$, if a key x is in the list L_i, it is also in L_{i-1}. Further there are up and down pointers between the x in L_i and the x in L_{i-1}.
- All the head(tail) nodes from neighboring lists are interconnected

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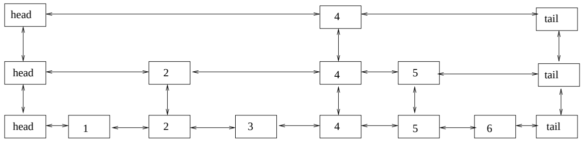
Example



Search

```
Search(k){
  pLeft = L_x.head;
  for (i=x;i>=0;i--){
    Search from pLeft in L_i to get the rightmost elem, r,
      with value <= k;
    pLeft = pointer to r in L_(i-1);
  }
  if (pLeft==k)
    return pLeft
  else
    return nil
  }
}
```