Annihilator Operators

We define three basic operations we can perform on this sequence:

1. Multiply the sequence by a constant: \( cA = \langle ca_0, ca_1, ca_2, \ldots \rangle \)
2. Shift the sequence to the left: \( L \hat{A} = \langle a_1, a_2, a_3, \ldots \rangle \)
3. Add two sequences: if \( A = \langle a_0, a_1, a_2, \ldots \rangle \) and \( B = \langle b_0, b_1, b_2, \ldots \rangle \), then \( A + B = \langle a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots \rangle \)

Example

- Consider the recurrence \( T(n) = 2T(n-1) \), \( T(0) = 1 \)
- If we solve for the first few terms of this sequence, we can see they are \( \langle 2^0, 2^1, 2^2, 2^3, \ldots \rangle \)
- Thus this recurrence becomes the sequence:
  \[ T = \langle 2^0, 2^1, 2^2, 2^3, \ldots \rangle \]

Example (II)

Let’s annihilate \( T = \langle 2^0, 2^1, 2^2, 2^3, \ldots \rangle \)

- Multiplying by a constant \( c = 2 \) gets:
  \[ 2T = \langle 2 \times 2^0, 2 \times 2^1, 2 \times 2^2, 2 \times 2^3, \ldots \rangle = \langle 2^1, 2^2, 2^3, 2^4, \ldots \rangle \]
- Shifting one place to the left gets \( LT = \langle 2^1, 2^2, 2^3, 2^4, \ldots \rangle \)
- Adding the sequence \( LT \) and \( -2T \) gives:
  \[ LT - 2T = \langle 2^1 - 2^1, 2^2 - 2^2, 2^3 - 2^3, \ldots \rangle = \langle 0, 0, 0, \ldots \rangle \]
Distributive Property

- The distributive property holds for these three operators
- Thus can rewrite $LT - 2T$ as $(L - 2)T$
- The operator $(L - 2)$ annihilates $T$ (makes it the sequence of all 0's)
- Thus $(L - 2)$ is called the annihilator of $T$

0, the “Forbidden Annihilator”

- Multiplication by 0 will annihilate any sequence
- Thus we disallow multiplication by 0 as an operation
- In particular, we disallow $(c - c) = 0$ for any $c$ as an annihilator
- Must always have at least one $L$ operator in any annihilator!

Uniqueness

- An annihilator annihilates exactly one type of sequence
- In general, the annihilator $L - c$ annihilates any sequence of the form $(a_0c^n)$
- If we find the annihilator, we can find the type of sequence, and thus solve the recurrence
- We will need to use the base case for the recurrence to solve for the constant $a_0$

Uniqueness

- The last example implies that an annihilator annihilates one type of sequence, but does not annihilate other types of sequences
- Thus Annihilators can help us classify sequences, and thereby solve recurrences
**Lookup Table**

- The annihilator \( L - a \) annihilates any sequence of the form \((c_1a^n)\)

**Example**

First calculate the annihilator:

- Recurrence: \( T(n) = 4 \times T(n - 1), \ T(0) = 2 \)
- Sequence: \( T = (2, 2 \times 4, 2 \times 4^2, 2 \times 4^3, \ldots) \)
- Calculate the annihilator:
  - \( LT = (2 \times 4, 2 \times 4^2, 2 \times 4^3, 2 \times 4^4, \ldots) \)
  - \( 4T = (2 \times 4, 2 \times 4^2, 2 \times 4^3, 2 \times 4^4, \ldots) \)
  - Thus \( LT - 4T = (0, 0, 0, 0, \ldots) \)
  - And so \( L - 4 \) is the annihilator

**Example (II)**

Now use the annihilator to solve the recurrence

- Look up the annihilator in the "Lookup Table"
- It says: "The annihilator \( L - 4 \) annihilates any sequence of the form \((c_14^n)\)"
- Thus \( T(n) = c_14^n \), but what is \( c_1 \)?
- We know \( T(0) = 2 \), so \( T(0) = c_14^0 = 2 \) and so \( c_1 = 2 \)
- Thus \( T(n) = 2 \times 4^n \)

**Multiple Operators**

- We can apply multiple operators to a sequence
- For example, we can multiply by the constant \( c \) and then by the constant \( d \) to get the operator \( cd \)
- We can also multiply by \( c \) and then shift left to get \( cLT \) which is the same as \( LcT \)
- We can also shift the sequence twice to the left to get \( L^2T \)

**In Class Exercise**

Consider the recurrence \( T(n) = 3 \times T(n - 1), \ T(0) = 3 \),

- Q1: Calculate \( T(0), T(1), T(2) \) and \( T(3) \) and write out the sequence \( T \)
- Q2: Calculate \( LT \), and use it to compute the annihilator of \( T \)
- Q3: Look up this annihilator in the lookup table to get the general solution of the recurrence for \( T(n) \)
- Q4: Now use the base case \( T(0) = 3 \) to solve for the constants in the general solution

**Multiple Operators**

- We can string operators together to annihilate more complicated sequences
- Consider: \( T = (2^0 + 3^0, 2^1 + 3^1, 2^2 + 3^2, \ldots) \)
- We know that \( (L - 2) \) annihilates the powers of 2 while leaving the powers of 3 essentially untouched
- Similarly, \( (L - 3) \) annihilates the powers of 3 while leaving the powers of 2 essentially untouched
- Thus if we apply both operators, we’ll see that \((L - 2)(L - 3)\) annihilates the sequence \( T \)
The Details

- Consider: \( T = (a^0 + b^0, a^1 + b^1, a^2 + b^2, \ldots) \)
- \( LT = (a^1 + b^1, a^2 + b^2, a^3 + b^3, \ldots) \)
- \( aT = (a^1 + a \cdot b^0, a^2 + a \cdot b^1, a^3 + a \cdot b^2, \ldots) \)
- \( LT - aT = ((b - a)b^0, (b - a)b^1, (b - a)b^2, \ldots) \)
- We know that \((L - a)T\) annihilates the \(a\) terms and multiplies the \(b\) terms by \(b - a\)
- Thus \((L - a)T = ((b - a)b^0, (b - a)b^1, \ldots)\)
- And so the sequence \((L - a)T\) is annihilated by \((L - b)\)
- Thus the annihilator of \(T\) is \((L - b)(L - a)\)

Factoring

- In general, the annihilator \((L - a)(L - b)\) (where \(a \neq b\)) will annihilate only all sequences of the form \((c_1a^n + c_2b^n)\)
- We will often multiply out \((L - a)(L - b)\) to \(L^2 - (a + b)L + ab\)
- Left as an exercise to show that \((L - a)(L - b)T\) is the same as \((L^2 - (a + b)L + ab)T\)
- \(L^2 - L - 1\) is an annihilator that is not in our lookup table
- However, we can factor this annihilator (using the quadratic formula) to get something similar to what’s in the lookup table
- \(L^2 - L - 1 = (L - \phi)(L - \bar{\phi})\), where \(\phi = \frac{1 + \sqrt{5}}{2}\) and \(\bar{\phi} = \frac{1 - \sqrt{5}}{2}\).

Lookup Table

- The annihilator \(L - a\) annihilates sequences of the form \((c_1a^n)\)
- The annihilator \((L - a)(L - b)\) (where \(a \neq b\)) annihilates sequences of the form \((c_1a^n + c_2b^n)\)

Quadratic Equation

"Me fail English? That's Unpossible!" - Ralph, the Simpsons

High School Algebra Review:

- To factor something of the form \(ax^2 + bx + c\), we use the Quadratic Formula:
- \(ax^2 + bx + c\) factors into \((x - \phi)(x - \bar{\phi})\), where:
  \[ \phi = \frac{b + \sqrt{b^2 - 4ac}}{2a} \]  \[ \bar{\phi} = \frac{b - \sqrt{b^2 - 4ac}}{2a} \]
Example

- To factor: $L^2 - L - 1$
- Rewrite: $1 \cdot L^2 - 1 \cdot L - 1$, $a = 1$, $b = -1$, $c = -1$
- From Quadratic Formula: $\phi = \frac{1 + \sqrt{5}}{2}$ and $\bar{\phi} = \frac{1 - \sqrt{5}}{2}$
- So $L^2 - L - 1$ factors to $(L - \phi)(L - \bar{\phi})$

The Punchline

- Recall Fibonacci recurrence: $T(0) = 0$, $T(1) = 1$, and $T(n) = T(n - 1) + T(n - 2)$
- The final explicit formula for $T(n)$ is thus:

$$T(n) = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

(Amazingly, $T(n)$ is always an integer, in spite of all of the square roots in its formula.)

Back to Fibonacci

- Recall the Fibonacci recurrence is $T(0) = 0$, $T(1) = 1$, and $T(n) = T(n - 1) + T(n - 2)$
- We’ve shown the annihilator for $T$ is $(L - \phi)(L - \bar{\phi})$, where $\phi = \frac{1 + \sqrt{5}}{2}$ and $\bar{\phi} = \frac{1 - \sqrt{5}}{2}$
- If we look this up in the “Lookup Table”, we see that the sequence $T$ must be of the form $(c_1 \phi^n + c_2 \bar{\phi}^n)$
- All we have left to do is solve for the constants $c_1$ and $c_2$
- Can use the base cases to solve for these

Finding the Constants

- We know $T = (c_1 \phi^n + c_2 \bar{\phi}^n)$, where $\phi = \frac{1 + \sqrt{5}}{2}$ and $\bar{\phi} = \frac{1 - \sqrt{5}}{2}$
- We know

$$T(0) = c_1 + c_2 = 0 \quad (6)$$
$$T(1) = c_1 \phi + c_2 \bar{\phi} = 1 \quad (7)$$

- We’ve got two equations and two unknowns
- Can solve to get $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$.

Todo

- Finish hw2!