CS 361, Lecture 2

Jared Saia
University of New Mexico

Today's Outline

- Asymptotic Analysis

Questions from last time

Express the following in O notation

- $n^3/1000 - 100n^2 - 100n + 3$
- $\log n + 100$
- $10 * \log^2 n + 100$
- $\sum_{i=1}^{n} i$

Computing big-O of an Algorithm

- Write down a formula, $f(n)$, which gives the number of elementary operations performed by the algorithm as a function of the input size, n
- Compute the big-O value for $f(n)$

An Example

Consider the following (silly) algorithm:

Alg 1 (int n)

```
For i=1 to n
  For j=1 to i
    print "hi"
```


An Example (II)

- First we write down the formula f giving the number of basic operations the algorithm performs: $f = \sum_{i=1}^{n} i = (n+1)n/2$
- Next we compute the big-O value for f: $(n+1)n/2$ is $O(n^2)$

We can then say that Alg1 takes $O(n^2)$ time. Or, for short, we just say Alg1 is $O(n^2)$

Examples from last class

Following are some formulas that represent the number of operations of some algorithm. Give the big-O notation for each.

- E.g. n, $10,000n - 2000$, and $.5n + 2$ are all $O(n)$
- $n + \log n$, $n - \sqrt{n}$ are $O(n)$
- $n^2 + n + \log n$, $10n^2 + n - \sqrt{n}$ are $O(n^2)$
- $n \log n + 10n$ is $O(n \log n)$
- $10 * \log^2 n$ is $O(\log^2 n)$
- $n\sqrt{n} + n \log n + 10n$ is $O(n\sqrt{n})$
- $10,000$, 2^{50} and 4 are $O(1)$

Computing big-O of an Algorithm

Following is a shorter way to compute big-O for an algorithm:

"Atomic operations"	Constant time
Consecutive statements	Sum of times
Conditionals	Larger branch time plus test time
Loops	Sum of iterations
Function Calls	Time of function body
Recursive Functions	Solve Recurrence Relation

Alg: Linear Search

```
bool LinearSearch (int arr[], int n, int key){
  for (int i=0;i<n;i++){
    if (arr[i]==key)
      return true;
  }
  return false;
}
```


Alg: Binary Search

```
bool BinarySearch (int arr[], int s, int e, int key){
  if (e-s<=0) return false;
  int mid = (e-s)/2;
  if (arr[key]==arr[mid]){
    return true;
  }else if (key < arr[mid]){
    return BinarySearch (arr,s,mid,key);}
  else{
    return BinarySearch (arr,mid,e,key)}
}
```


Linear Search Analysis

- To analyze the linear search algorithm, we consider the worst case
- The worst case occurs when the key is the very last element in the array
- In this case, the algorithm takes $O(n)$ time
- Thus we say that the run time of Linear Search is $O(n)$
- (Note that the average time of Linear Search is also $O(n)$)

Binary Search Analysis

- Note that even in the worst case, the size of the array we search is being split in half in each call
- Thus if x is the number of recursive calls, and n is the original size of the array, $n(1/2)^x = 1$ in the worst case
- This implies that $2^x = n$
- Taking log of both sides, we get $x = \log n$, which means that there are $\log n$ recursive calls in the worst case
- Since each invocation of the function takes $O(1)$ time (minus the recursive calls), and the total number of invocations is at most $\log n$, the running time is $O(\log n)$

Comparison

- Linear Search is $O(n)$ time
- Binary Search is $O(\log n)$ time
- Binary Search is a *much* faster algorithm, particularly for large input sizes

A digression on logs

It rolls down stairs alone or in pairs,
and over your neighbor's dog,
it's great for a snack or to put on your back,
it's log, log, log!
\- *"The Log Song" from the Ren and Stimpy Show*

- The log function shows up very frequently in algorithm analysis
- As computer scientists, when we use log, we'll mean $\log_2$ (i.e. if no base is given, assume base 2)

Definition

- $\log_x y$ is by definition the value z such that $x^z = y$
- $x^{\log_x y} = y$ by definition

Examples

- $\log 1 = 0$
- $\log 2 = 1$
- $\log 32 = 5$
- $\log 2^k = k$

Note: $\log n$ is way, way smaller than n for large values of n

Examples

- $\log_3 9 = 2$
- $\log_5 125 = 3$
- $\log_4 16 = 2$
- $\log_{24} 24^{100} = 100$

Facts about exponents

Recall that:

- $(x^y)^z = x^{yz}$
- $x^y x^z = x^{y+z}$

From these, we can derive some facts about logs

Facts about logs

To prove both equations, raise both sides to the power of 2, and use facts about exponents

- Fact 1: $\log(xy) = \log x + \log y$
- Fact 2: $\log a^c = c \log a$

Incredibly useful fact about logs

- Fact 3: $\log_c a = \log a / \log c$

To prove this, consider the equation $a = c^{\log_c a}$, take $\log_2$ of both sides, and use Fact 2.

Log facts to memorize

Memorize these facts

- Fact 1: $\log(xy) = \log x + \log y$
- Fact 2: $\log a^c = c \log a$
- Fact 3: $\log_c a = \log a / \log c$

These facts are sufficient for all your logarithm needs. (You just need to figure out how to use them)

Logs and O notation

- Note that $\log_8 n = \log n / \log 8$.
- Note that $\log_{600} n^{200} = 200 * \log n / \log 600$.
- Note that $\log_{100000} 30*n^2 = 2*\log n/\log 100000 + \log 30/\log 100000$.
- Thus, $\log_8 n$, $\log_{600} n^{600}$, and $\log_{100000} 30*n^2$ are all $O(\log n)$
- In general, for any constants k_1 and k_2, $\log_{k_1} n^{k_2} = k_2 \log n / \log k_1$, which is just $O(\log n)$

Take Away

- All log functions of form $k_1 \log_{k_2} k_3 * n^{k_4}$ for constants k_1, k_2, k_3 and k_4 are $O(\log n)$
- For this reason, we don't really "care" about the base of the log function when we do asymptotic notation
- Thus, binary search, ternary search and k-ary search all take $O(\log n)$ time

Important Note

- $\log^2 n = (\log n)^2$
- $\log^2 n$ is $O(\log^2 n)$, *not* $O(\log n)$
- This is true since $\log^2 n$ grows asymptotically faster than $\log n$
- All log functions of form $k_1 \log_{k_3}^{k_2} k_4 * n^{k_5}$ for constants k_1, k_2, k_3,k_4 and k_5 are $O(\log^{k_2} n)$

In-Class Exercise

Simplify and give O notation for the following functions. In the big-O notation, write all logs base 2:

- $\log 10n^2$
- $\log_5(n/4)$
- $\log^2 n^4$
- $2^{\log_4 n}$
- $\log\log\sqrt{n}$

Does big-O really matter?

Let $n = 100000$ and $\Delta t = 1\mu$s

$\log n$	$1.2 * 10^{-5}$ seconds
$\sqrt{n}$	$3.2 * 10^{-4}$ seconds
n	.1 seconds
$n\log n$	1.2 seconds
$n\sqrt{n}$	31.6 seconds
n^2	2.8 hours
n^3	31.7 years
2^n	$>$ 1 century

(from Classic Data Structures in C++ by Timothy Budd)

Another Interview Question

- The Question: Design an algorithm to return the largest sum of contiguous integers in an array of ints
- Example: if the input is $(-10, 2, 3, -2, 0, 5, -15)$, the largest sum is 8, which we get from $(2, 3, -2, 0, 5)$.

A Naive Algorithm

```
MaxSeq1 (int arr[], int n)
  int max = 0;
  for (int i = 0;i<n;i++)
    for (int j=i;j<n;j++)
      int sum = 0;
      for (int k=i;k<=j;k++)
        sum += arr[k];
        if (sum > max)
          max = sum;
  return max;
```

Analysis

- Need to count the total number of operations of MaxSeq1
- Might as well assume time to do the inner loop is 1 (since it's a constant and therefore $O(1)$)
- Let $f(n)$ be the runtime for an array of size n

$$
\begin{align}
f(n) &= \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=i}^{j} 1 \tag{1}\\
&= \sum_{i=1}^{n}\sum_{j=i}^{n}(j-i) \tag{2}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n-i} j \tag{3}\\
&= \sum_{i=1}^{n}((n-i)/2)(n-i+1) \tag{4}
\end{align}
$$

Analysis (II)

$$
\begin{align}
f(n) &= \sum_{i=1}^{n}((n-i)/2)(n-i+1) \tag{5}\\
&= \sum_{i=1}^{n-1}(i/2)(i+1) \tag{6}\\
&= 1/2 * \sum_{i=1}^{n-1}(i^2+i) \tag{7}\\
&= 1/2 * (\sum_{i=1}^{n-1} i^2 + \sum_{i=1}^{n-1} i) \tag{8}\\
&= 1/2 * (O(n^3) + O(n^2)) \tag{9}\\
&= O(n^3) \tag{10}
\end{align}
$$

Challenge

- MaxSeq1 is very slow
- This kind of algorithm won't impress an interviewer
- Can you do better?

Todo

- Finish pretest, due next Tuesday!
- Sign up for the class mailing list (cs361)
- Read Chapter 3 (Growth of Functions) in textbook