Outline

- Annihilators
- Transformations

Anihilator Method

- Write down the annihilator for the recurrence
- Factor the annihilator
- Look up the factored annihilator in the “Lookup Table” to get general solution
- Solve for constants of the general solution by using initial conditions

Lookup Table

$$(L - a_0)^{b_0}(L - a_1)^{b_1} \ldots (L - a_k)^{b_k}$$

(where $a_i \neq a_j$, for $i \neq j$) annihilates only sequences of the form:

$$\langle p_1(n)a_0^n + p_2(n)a_1^n + \ldots p_k(n)a_k^n \rangle$$

where $p_i(n)$ is a polynomial of degree $b_i - 1$
Examples

- Q: What does \((L - 3)(L - 2)(L - 1)\) annihilate?
  - A: \(c_01^n + c_12^n + c_23^n\)

- Q: What does \((L - 3)^2(L - 2)(L - 1)\) annihilate?
  - A: \(c_01^n + c_12^n + (c_2n + c_3)3^n\)

- Q: What does \((L - 1)^4\) annihilate?
  - A: \((c_0n^3 + c_1n^2 + c_2n + c_3)1^n\)

- Q: What does \((L - 1)^3(L - 2)^2\) annihilate?
  - A: \((c_0n^2 + c_1n + c_2)1^n + (c_3n + c_4)2^n\)

Example 1

Consider the recurrence \(T(n) = 2T(n - 1) - T(n - 2), T(0) = 0, T(1) = 1\). Note that:

\[
L^2T = \langle T_{n+2} \rangle \\
= \langle 2T_{n+1} - T_n \rangle \\
LT = \langle T_{n+1} \rangle \\
T = \langle T_n \rangle 
\]

- Thus \(L^2T - 2LT + T = \langle 0 \rangle\)
- So \(L^2 - 2L + 1\) is the annihilator of \(T\)

Example (II)

Consider the recurrence \(T(n) = 7T(n - 1) - 16T(n - 2) + 12T(n - 3), T(0) = 1, T(1) = 5, T(2) = 17\). Note that:

\[
L^3T = \langle T_{n+3} \rangle \\
= \langle 7T_{n+2} - 16T_{n+1} + 12T_n \rangle \\
L^2T = \langle T_{n+2} \rangle \\
LT = \langle T_{n+1} \rangle \\
T = \langle T_n \rangle 
\]

- Thus \(L^3T - 7L^2T + 16LT - 12T = \langle 0 \rangle\)
- So the annihilator is \(L^3 - 7L^2 + 16L - 12\)
Example (II)

Consider the recurrence $T(n) = 7T(n-1) - 16T(n-2) + 12T(n-3)$, $T(0) = 1$, $T(1) = 5$, $T(2) = 17$

- **Write down the annihilator:** From the definition of the sequence, we can see that $L^3T - 7L^2T + 16LT - 12T = 0$, so the annihilator is $L^3 - 7L^2 + 16L - 12$
- **Factor the annihilator:** We can factor by hand or using a computer program to get $L^3 - 7L^2 + 16L - 12 = (L - 2)(L - 3)^2$
- **Look up to get general solution:** The annihilator $(L - 2)(L - 3)^2$ annihilates sequences of the form $(c_0n + c_12^n + c_23^n)$
- **Solve for constants:** $T(0) = 1 = c_1 + c_2$, $T(1) = 5 = 2c_0 + 2c_1 + 3c_2$, $T(2) = 17 = 8c_0 + 4c_1 + 9c_2$. We’ve got three equations and three unknowns. Solving by hand, we get that $c_0 = 1, c_1 = 0, c_2 = 1$. **Thus:** $T(n) = n2^n + 3^n$

At Home Exercise

Consider the recurrence $T(n) = 6T(n-1) - 9T(n-2)$, $T(0) = 1$, $T(1) = 6$

- Q1: What is the annihilator of this sequence?
- Q2: What is the factored version of the annihilator?
- Q3: What is the general solution for the recurrence?
- Q4: What are the constants in this general solution?

(Note: You can check that your general solution works for $T(2)$)

Example

- In a **height-balanced tree**, the height of two subtrees of any node differ by at most one
- Let $T(n)$ be the smallest number of nodes needed to obtain a height balanced binary tree of height $n$
- Q: What is a recurrence for $T(n)$?
- A: Divide this into smaller subproblems
  - To get a height-balanced tree of height $n$ with the smallest number of nodes, need one subtree of height $n-1$, and one of height $n-2$, plus a root node
  - Thus $T(n) = T(n-1) + T(n-2) + 1$
Example

- Let’s solve this recurrence: \( T(n) = T(n-1) + T(n-2) + 1 \)
  (Let \( T_n = T(n) \), and \( T = \langle T_n \rangle \))
- We know that \((L^2-L-1)\) annihilates the homogeneous terms
- Let’s apply it to the entire equation:
  \[
  (L^2-L-1) \langle T_n \rangle = L^2 \langle T_n \rangle - L \langle T_n \rangle - 1 \langle T_n \rangle = \langle T_{n+2} \rangle - \langle T_{n+1} \rangle - \langle T_n \rangle = \langle T_{n+2} - T_{n+1} - T_n \rangle = \langle 1, 1, 1, \cdots \rangle
  \]

Example

- This is close to what we want but we still need to annihilate \(\langle 1, 1, 1, \cdots \rangle\)
- It’s easy to see that \(L - 1\) annihilates \(\langle 1, 1, 1, \cdots \rangle\)
- Thus \((L^2-L-1)(L-1)\) annihilates \(T(n) = T(n-1) + T(n-2) + 1\)
- When we factor, we get \((L-\phi)(L-\bar{\phi})(L-1),\) where \(\phi = \frac{1+\sqrt{5}}{2}\) and \(\bar{\phi} = \frac{1-\sqrt{5}}{2}\).

General Rule

To find the annihilator for recurrences with non-homogeneous terms, do the following:

- Find the annihilator \(a_1\) for the homogeneous part
- Find the annihilator \(a_2\) for the non-homogeneous part
- The annihilator for the whole recurrence is then \(a_1a_2\)

Lookup

- Looking up \((L - \phi)(L - \bar{\phi})(L - 1)\) in the table
- We get \(T(n) = c_1\phi^n + c_2\bar{\phi}^n + c_31^n\)
- If we plug in the appropriate initial conditions, we can solve for these three constants
- We’ll need to get equations for \(T(2)\) in addition to \(T(0)\) and \(T(1)\)
Another Example

- Consider $T(n) = T(n-1) + T(n-2) + 2$.
- The residue is $\langle 2, 2, 2, \cdots \rangle$ and
- The annihilator is still $(L^2 - L - 1)(L - 1)$, but the equation for $T(2)$ changes!

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + n$.
- The residue is $\langle 1, 2, 3, 4, \cdots \rangle$.
- The annihilator is now $(L^2 - L - 1)(L - 1)^2$.

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + 2^n$.
- The residue is $\langle 1, 2, 4, 8, \cdots \rangle$ and
- The annihilator is now $(L^2 - L - 1)(L - 2)$.

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + n^2$.
- The residue is $\langle 1, 4, 9, 16, \cdots \rangle$ and
- The annihilator is $(L^2 - L - 1)(L - 1)^3$. 
**Another Example**

- Consider $T(n) = T(n - 1) + T(n - 2) + n^2 - 2^n$.
- The residue is $\{1 - 1, 4 - 4, 9 - 8, 16 - 16, \ldots\}$ and the annihilation is $(L^2 - L - 1)(L - 1)^3(L - 2)$.

**Limitations**

- Our method does not work on $T(n) = T(n - 1) + \frac{1}{n}$ or $T(n) = T(n - 1) + \log n$.
- The problem is that $\frac{1}{n}$ and $\log n$ do not have annihilators.
- Our tool, as it stands, is limited.
- Key idea for strengthening it is transformations.

**In Class Exercise**

- Consider $T(n) = 3 \cdot T(n - 1) + 3^n$.
- Q1: What is the homogeneous part, and what annihilates it?
- Q2: What is the non-homogeneous part, and what annihilates it?
- Q3: What is the annihilator of $T(n)$, and what is the general form of the recurrence?

**Transformations Idea**

- Consider the recurrence giving the run time of mergesort $T(n) = 2T(n/2) + kn$ (for some constant $k$), $T(1) = 1$.
- How do we solve this?
- We have no technique for annihilating terms like $T(n/2)$.
- However, we can transform the recurrence into one with which we can work.
Transformation

- Let \( n = 2^i \) and rewrite \( T(n) \):
- \( T(2^0) = 1 \) and \( T(2^i) = 2T(\frac{2^i}{2}) + k2^i = 2T(2^{i-1}) + k2^i \)
- Now define a new sequence \( t \) as follows: \( t(i) = T(2^i) \)
- Then \( t(0) = 1, t(i) = 2t(i-1) + k2^i \)

Reverse Transformation

- We've got a solution for \( t(i) \) and we want to transform this into a solution for \( T(n) \)
- Recall that \( t(i) = T(2^i) \) and \( 2^i = n \)
  \[
  t(i) = (c_1i + c_2)2^i \quad (1)
  \]
  \[
  T(2^i) = (c_1i + c_2)2^i \quad (2)
  \]
  \[
  T(n) = (c_1 \lg n + c_2)n \quad (3)
  \]
  \[
  = c_1n \lg n + c_2n \quad (4)
  \]
  \[
  = O(n \lg n) \quad (5)
  \]

Now Solve

- We've got a new recurrence: \( t(0) = 1, t(i) = 2t(i-1) + k2^i \)
- We can easily find the annihilator for this recurrence
- \((L - 2)\) annihilates the homogeneous part, \((L - 2)\) annihilates the non-homogeneous part, So \((L - 2)(L - 2)\) annihilates \( t(i) \)
- Thus \( t(i) = (c_1i + c_2)2^i \)

Success!

- We could not find the annihilator of \( T(n) \) so:
- We did a transformation to a recurrence we could solve, \( t(i) \)
  (we let \( n = 2^i \) and \( t(i) = T(2^i) \))
- We found the annihilator for \( t(i) \), and solved the recurrence for \( t(i) \)
- We reverse transformed the solution for \( t(i) \) back to a solution for \( T(n) \)
Another Example

- Consider the recurrence $T(n) = 9T\left(\frac{n}{3}\right) + kn$, where $T(1) = 1$ and $k$ is some constant.
- Let $n = 3^i$ and rewrite $T(n)$:
  - $T(2^0) = 1$ and $T(3^i) = 9T(3^{i-1}) + k3^i$
- Now define a sequence $t$ as follows $t(i) = T(3^i)$
- Then $t(0) = 1$, $t(i) = 9t(i - 1) + k3^i$

Reverse Transformation

- $t(i) = c_19^i + c_23^i$
- Recall: $t(i) = T(3^i)$ and $3^i = n$
  
  \[
  t(i) = c_19^i + c_23^i \\
  T(3^i) = c_19^i + c_23^i \\
  T(n) = c_1(3^i)^2 + c_23^i \\
  = c_1n^2 + c_2n \\
  = \Theta(n^2)
  \]

Now Solve

- $t(0) = 1$, $t(i) = 9t(i - 1) + k3^i$
- This is annihilated by $(L - 9)(L - 3)$
- So $t(i)$ is of the form $t(i) = c_19^i + c_23^i$

In Class Exercise

- Consider the recurrence $T(n) = 2T(n/4) + kn$, where $T(1) = 1$, and $k$ is some constant

  - Q1: What is the transformed recurrence $t(i)$? How do we rewrite $n$ and $T(n)$ to get this sequence?
  - Q2: What is the annihilator of $t(i)$? What is the solution for the recurrence $t(i)$?
  - Q3: What is the solution for $T(n)$? (i.e. do the reverse transformation)
A Final Example

Not always obvious what sort of transformation to do:

- Consider \( T(n) = 2T(\sqrt{n}) + \log n \)
- Let \( n = 2^i \) and rewrite \( T(n) \):
  - \( T(2^i) = 2T(2^{i/2}) + i \)
- Define \( t(i) = T(2^i) \):
  - \( t(i) = 2t(i/2) + i \)

This final recurrence is something we know how to solve!
- \( t(i) = O(i \log i) \)
- The reverse transform gives:
  - \( t(i) = O(i \log i) \) \hspace{1cm} (6)
  - \( T(2^i) = O(i \log i) \) \hspace{1cm} (7)
  - \( T(n) = O(\log n \log \log n) \) \hspace{1cm} (8)