CS 362, Lecture 10

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Today's Outline

- Fractional Knapsack Wrapup
- Amortized Analysis

Proof

- Assume the objects are sorted in order of cost per pound. Let v_i be the value for item i and let w_i be its weight.
- Let x_i be the *fraction* of object i selected by greedy and let V be the total value obtained by greedy
- Consider some arbitrary solution, B, and let x'_i be the fraction of object i taken in B and let V' be the total profit obtained by B
- We want to show that $V' \leq V$ or that $V - V' \geq 0$

Proof

- Let k be the smallest index with $x_k < 1$
- Note that for $i \leq k$, $x_i = 1$ and for $i > k$, $x_i = 0$
- You will show that for all i,

$$(x_i - x'_i)\frac{v_i}{w_i} \geq (x_i - x'_i)\frac{v_k}{w_k}$$

Proof

$$
\begin{aligned}
V - V' &= \sum_{i=1}^{n} x_i v_i - \sum_{i=1}^{n} x_i' v_i && (1)\\
&= \sum_{i=1}^{n} (x_i - x_i') * v_i && (2)\\
&= \sum_{i=1}^{n} (x_i - x_i') * w_i \left(\frac{v_i}{w_i}\right) && (3)\\
&\geq \sum_{i=1}^{n} (x_i - x_i') * w_i \left(\frac{v_k}{w_k}\right) && (4)\\
&\geq \left(\frac{v_k}{w_k}\right) * \sum_{i=1}^{n} (x_i - x_i') * w_i && (5)\\
&\geq 0 && (6)
\end{aligned}
$$

Proof

- Note that the last step follows because $\frac{v_k}{w_k}$ is positive and because:

$$
\begin{aligned}
\sum_{i=1}^{n} (x_i - x_i') * w_i &= \sum_{i=1}^{n} x_i w_i - \sum_{i=1}^{n} x_i' * w_i && (7)\\
&= W - W' && (8)\\
&\geq 0. && (9)
\end{aligned}
$$

- Where W is the total weight taken by greedy and W' is the total weight for the strategy B
- We know that $W \geq W'$

In-Class Exercise

Consider the inequality:

$$(x_i - x_i')\frac{v_i}{w_i} \geq (x_i - x_i')\frac{v_k}{w_k}$$

- Q1: Show this inequality is true for $i < k$
- Q2: Show it's true for $i = k$
- Q3: Show it's true for $i > k$

Q1

$$(x_i - x_i')\frac{v_i}{w_i} \geq (x_i - x_i')\frac{v_k}{w_k}$$

- Q1: Show that the inequality is true for $i < k$
- For $i < k$, $(x_i - x_i') \geq 0$
- If $(x_i - x_i') = 0$, trivially true. Otherwise, can divide both sides of the inequality by $x_i - x_i'$ to get

$$\frac{v_i}{w_i} \geq \frac{v_k}{w_k}.$$

- This is true since the items are sorted by profit per weight

Q2

$$(x_i - x'_i)\frac{v_i}{w_i} \geq (x_i - x'_i)\frac{v_k}{w_k}$$

- Q2: Show that the inequality is true for $i = k$
- When $i = k$, we have

$$(x_k - x'_k)\frac{v_k}{w_k} \geq (x_k - x'_k)\frac{v_k}{w_k}$$

- Which is true since the left side equals the right side

Q3

$$(x_i - x'_i)\frac{v_i}{w_i} \geq (x_i - x'_i)\frac{v_k}{w_k}$$

- Q3: Show that the inequality is true for $i > k$
- For $i < k$, $(x_i - x'_i) \leq 0$
- If $(x_i - x'_i) = 0$, trivially true. Otherwise can divide both sides of the inequality by $x_i - x'_i$ to get

$$\frac{v_i}{w_i} \leq \frac{v_k}{w_k}.$$

- This is obviously true since the items are sorted by profit per weight
- *Note that the direction of the inequality changed when we divided by $(x_i - x'_i)$, since it is negative*

Amortized Analysis

"I will gladly pay you Tuesday for a hamburger today" - Wellington Wimpy

- In amortized analysis, time required to perform a sequence of data structure operations is averaged over all the operations performed
- Typically used to show that the average cost of an operation is small for a sequence of operations, even though a single operation can cost a lot

Amortized analysis

Amortized analysis is *not* average case analysis.

- *Average Case Analysis*: the expected cost of each operation
- *Amortized analysis*: the average cost of each operation *in the worst case*
- Probability is not involved in amortized analysis

Types of Amortized Analysis

- *Aggregate Analysis*
- *Accounting or Taxation Method*
- *Potential method*
- We'll see each method used for 1) a stack with the additional operation MULTIPOP and 2) a binary counter

Aggregate Analysis

- We get an upperbound $T(n)$ on the total cost of a sequence of n operations. The average cost per operation is then $T(n)/n$, which is also the amortized cost per operation

Stack with Multipop

- Recall that a standard stack has the operations PUSH and POP
- Each of these operations runs in $O(1)$ time, so let's say the cost of each is 1
- Now for a stack S and number k, let's add the operation MULTIPOP which removes the top k objects on the stack
- Multipop just calls Pop either k times or until the stack is empty

Multipop

- Q: What is the running time of Multipop(S,k) on a stack of s objects?
- A: The cost is min(s,k) pop operations
- If there are n stack operations, in the worst case, a single Multipop can take $O(n)$ time

Multipop Analysis

- Let's analyze a sequence of n push, pop, and multipop operations on an initially empty stack
- The worst case cost of a multipop operation is $O(n)$ since the stack size is at most n, so the worst case time for any operation is $O(n)$
- Hence a sequence of n operations costs $O(n^2)$

The Problem

- This analysis is technically correct, but overly pessimistic
- While some of the multipop operations can take $O(n)$ time, not all of them can
- We need some way to average over the entire sequence of n operations

Aggregate Analysis

- In fact, the total cost of n operations on an initially empty stack is $O(n)$
- Why? Because each object can be popped at most once for each time that it is pushed
- Hence the number of times POP (including calls within Multipop) can be called on a nonempty stack is at most the number of Push operations which is $O(n)$

Aggregate Analysis

- Hence for any value of n, any sequence of n Push, Pop, and Multipop operations on an initially empty stack takes $O(n)$ time
- The average cost of an operation is thus $O(n)/n = O(1)$
- Thus all stack operations have an *amortized* cost of $O(1)$

Another Example

Another example where we can use aggregate analysis:

- Consider the problem of creating a k bit binary counter that counts upward from 0
- We use an array $A[0..k-1]$ of bits as the counter
- A binary number x that is stored in A has its lowest-order bit in $A[0]$ and highest order bit in $A[k-1]$ $(x = \sum_{i=0}^{k-1} A[i] * 2^i)$

Binary Counter

- Initially $x = 0$ so $A[i] = 0$ for all $i = 0, 1, \ldots, k-1$
- To add 1 to the counter, we use a simple procedure which scans the bits from right to left, zeroing out 1's until it finally find a zero bit which it flips to a 1

Increment

```
Increment(A){
  i = 0;
  while(i<k && A[i]=1){
    A[i] = 0;
    i++;
  }
  if (i<k)
    A[i] = 1;
 }
```


Analysis

- It's not hard to see that in the worst case, the increment procedure takes time $\Theta(k)$
- Thus a sequence of n increments takes time $O(nk)$ in the worst case
- Note that again this bound is correct but overly pessimistic - not all bits flip each time increment is called!

Aggregate Analysis

- In fact, we can show that a sequence of n calls to Increment has a worst case time of $O(n)$
- $A[0]$ flips every time Increment is called, $A[1]$ flips over every other time, $A[2]$ flips over every fourth time, ...
- Thus if there are n calls to increment, $A[0]$ flips n times, $A[1]$ flips $\lfloor n/2 \rfloor$ times, $A[2]$ flips $\lfloor n/4 \rfloor$ times

Aggregate Analysis

- In general, for $i = 0, \ldots \lfloor \log n \rfloor$, bit $A[i]$ flips $\left\lfloor n/2^i \right\rfloor$ times in a sequence of n calls to Increment on an initially zero counter
- For $i > \lfloor \log n \rfloor$, bit $A[i]$ never flips
- Total number of flips in the sequence of n calls is thus

$$\sum_{i=0}^{\lfloor \log n \rfloor} \left\lfloor \frac{n}{2^i} \right\rfloor < n \sum_{i=0}^{\infty} \frac{1}{2^i} \quad (10)$$

$$= 2n \quad (11)$$

Aggregate Analysis

- Thus the worst-case time for a sequence of n Increment operations on an initially empty counter is $O(n)$
- The average cost of each operation in the worst case then is $O(n)/n = O(1)$

Accounting or Taxation Method

- The second method is called the accounting method in the book, but a better name might be the *taxation* method
- Suppose it costs us a dollar to do a Push or Pop
- We can then measure the run time of our algorithm in dollars (Time is money!)

Taxation Method for Multipop

- Instead of paying for each Push and Pop operation when they occur, let's tax the pushes to pay for the pops
- I.e. we tax the push operation 2 dollars, and the pop and multipop operations 0 dollars
- Then each time we do a push, we spend one dollar of the tax to pay for the push and then *save* the other dollar of the tax to pay for the inevitable pop or multipop of that item
- Note that if we do n operations, the total amount of taxes we collect is then $2n$

Taxation Method

- Like any good government (ha ha) we need to make sure that: 1) our taxes are low and 2) we can use our taxes to pay for all our costs
- We already know that our taxes for n operations are no more than $2n$ dollars
- We now want to show that we can use the 2 dollars we collect for each push to pay for all the push, pop and multipop operations

Taxation Method

- This is easy to show. When we do a push, we use 1 dollar of the tax to pay for the push and then store the extra dollar with the item that was just pushed on the stack
- Then all items on the stack will have one dollar stored with them
- Whenever we do a Pop, we can use the dollar stored with the item popped to pay for the cost of that Pop
- Moreover, whenever we do a Multipop, for each item that we pop off in the Multipop, we can use the dollar stored with that item to pay for the cost of popping that item

Taxation Method

- We've shown that we can use the 2 tax on each item pushed to pay for the cost of all pops, pushes and multipops.
- Moreover we know that this taxation scheme collects at most $2n$ dollars in taxes over n stack operations
- Hence we've shown that the amortized cost per operation is $O(1)$

Taxation Method for Binary Counter

- Let's now use the taxation method to show that the amortized cost of the Increment algorithm is $O(1)$
- Let's say that it costs us 1 dollar to flip a bit
- What is a good taxation scheme to ensure that we can pay for the costs of all flips but that we keep taxes low?

Taxation Scheme

- Let's tax the algorithm 2 dollars to set a bit to 1
- Now we need to show that: 1) this scheme has low total taxes and 2) we will collect enough taxes to pay for all of the bit flips
- Showing overall taxes are low is easy: Each time Increment is called, it sets at most one bit to a 1
- So we collect exactly 2 dollars in taxes each time increment is called
- Thus over n calls to Increment, we collect $2n$ dollars in taxes

Taxation Scheme

- We now need to show that our taxation scheme has enough money to pay for the costs of all operations
- This is easy: Each time we set a bit to a 1, we collect 2 dollars in tax. We use one dollar to pay for the cost of setting the bit to a 1, then we *store* the extra dollar on that bit
- When the bit gets flipped back from a 1 to a 0, we use the dollar already on that bit to pay for the cost of the flip!

Binary Counter

- We've shown that we can use the 2 tax each time a bit is set to a 1 to pay for all operations which flip a bit
- Moreover we know that this taxation scheme collects $2n$ dollars in taxes over n calls to Increment
- Hence we've shown that the amortized cost per call to Increment is $O(1)$

In Class Exercise

- A sequence of Pushes and Pops is performed on a stack whose size never exceeds k
- After every k operations, a copy of the entire stack is made for backup purposes
- Show that the cost of n stack operations, including copying the stack, is $O(n)$

In Class Exercise

- A sequence of Pushes and Pops is performed on a stack whose size never exceeds k
- After every k operations, a copy of the entire stack is made for backup purposes
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In Class Exercise

- Q1: What is your taxation scheme?
- Q2: What is the maximum amount of taxes this scheme collects over n operations?
- Q3: Show that your taxation scheme can pay for the costs of all operations