Today’s Outline

- Data Structures for Disjoint Sets

Disjoint Sets

- A disjoint set data structure maintains a collection \( \{S_1, S_2, \ldots, S_k\} \) of disjoint dynamic sets
- Each set is identified by a representative which is a member of that set
- Let’s call the members of the sets objects

Operations

We want to support the following operations:

- Make-Set(\( x \)): creates a new set whose only member (and representative) is \( x \)
- Union(\( x, y \)): unites the sets that contain \( x \) and \( y \) (call them \( S_x \) and \( S_y \)) into a new set that is \( S_x \cup S_y \). The new set is added to the data structure while \( S_x \) and \( S_y \) are deleted. The representative of the new set is any member of the set.
- Find-Set(\( x \)): Returns a pointer to the representative of the (unique) set containing \( x \)
Simple Union

Make-Set(x){
    parent(x) = x;
    size(x) = 1;
}

Simple-Union(x,y){
    xRep = Find-Set(x);
    yRep = Find-Set(y);
    if (size(xRep)) > size(yRep)){
        parent(yRep) = xRep;
    }else{
        parent(xRep) = yRep;
    }
    size(yRep) = size(yRep) + size(xRep);
}

Analysis

• We showed in last class that the heights of all trees are no
  more than logarithmic in the number of nodes in the tree
• Thus all of these operations take $O(\log n)$ time
• Q: Can we do better?
• A: Yes we can do much better in an amortized sense.

Shallow Threaded Trees

• One good idea is to just have every object keep a pointer to
  the leader of it’s set
• In other words, each set is represented by a tree of depth 1
• Then Make-Set and Find-Set are completely trivial, and they
  both take $O(1)$ time
• Q: What about the Union operation?

Union

• To do a union, we need to set all the leader pointers of one
  set to point to the leader of the other set
• To do this, we need a way to visit all the nodes in one of the
  sets
• We can do this easily by “threading” a linked list through
  each set starting with the sets leaders
• The threads of two sets can be merged by the Union algo-
  rithm in constant time
The Code

Make-Set(x){
    leader(x) = x;
    next(x) = NULL;
}
Find-Set(x){
    return leader(x);
}

Union(x,y){
    xRep = Find-Set(x);
    yRep = Find-Set(y);
    leader(yRep) = xRep;
    z = yRep;
    while(next(z)!=NULL){
        z = next(z);
        leader(z) = xRep;
    }
    next(z) = next(xRep);
    next(xRep) = yRep;
}

Example

Merging two sets stored as threaded trees. Bold arrows point to leaders; lighter arrows form the threads. Shaded nodes have a new leader.

Analysis

- Worst case time of Union is a constant times the size of the larger set
- So if we merge a one-element set with a $n$ element set, the run time can be $\Theta(n)$
- In the worst case, it's easy to see that $n$ operations can take $\Theta(n^2)$ time for this alg
Problem

- The main problem here is that in the worst case, we always get unlucky and choose to update the leader pointers of the larger set
- Instead let's purposefully choose to update the leader pointers of the smaller set
- To do this, we will need to keep track of the sizes of all the sets

The Code

```
Make-Weighted-Set(x){
    leader(x) = x;
    next(x) = NULL;
    size(x) = 1;
}
```

```
Weighted-Union(x,y){
    xRep = Find-Set(x);
    yRep = Find-Set(y)
    if(size(xRep)>size(yRep){
        Union(xRep,yRep);
        size(xRep) = size(xRep) + size(yRep);
    }else{
        Union(yRep,xRep);
        size(yRep) = size(xRep) + size(yRep);
    }
}
```

Analysis

- The Weighted-Union algorithm still takes $\Theta(n)$ time to merge two $n$ element sets
- However in an amortized sense, it is more efficient
- Intuitively, in order to merge two large sets, we need to perform a large number of cheap Weighted-Unions
- We will show that a sequence of $n$ Make-Weighted-Set operations and $m$ Weighted-Union operations takes $O(m+n\log n)$ time in the worst case.
Proof

- Whenever the leader of an object $x$ is changed by a call to Weighted-Union, the size of the set containing $x$ increases by a factor of at least 2.
- Thus if the leader of $x$ has changed $k$ times, the set containing $x$ has at least $2^k$ members.
- After the sequence of operations ends, the largest set has at most $n$ members.
- Thus the leader of any object $x$ has changed at most $\lceil \log n \rceil$ times.

Proof

- Let $n$ be the number of calls to Make-Weighted-Set and $m$ be the number of calls to Weighted-Union.
- We’ve shown that each of the objects that are not in singleton sets had at most $O(\log n)$ leader changes.
- Thus, the total amount of work done in updating the leader pointers is $O(n \log n)$.

Proof

- We’ve just shown that for $n$ calls to Make-Weighted-Set and $m$ calls to Weighted-Union, that total cost for updating leader pointers is $O(n \log n)$.
- We know that other than the work needed to update these leader pointers, each call to one of our functions does only constant work.
- Thus total amount of work is $O(n \log n + m)$.
- Thus each Weighted-Union call has amortized cost of $O(\log n)$.

Analysis

- Using Simple-Union, Find takes logarithmic worst case time and everything else is constant.
- Using Weighted-Union, Union takes logarithmic amortized time and everything else is constant.
- A third method allows us to get both of these operations in almost constant amortized time.

Side Note: We’ve just used the aggregate method of amortized analysis.
Path Compression

- We start with the unthreaded tree representation (from Simple-Union)
- Key Observation is that in any Find operation, once we get the leader of an object $x$, we can speed up future Find’s by redirecting $x$’s parent pointer directly to that leader
- We can also change the parent pointers of all ancestors of $x$ all the way up to the root (We’ll do this using recursion)
- This modification to Find is called path compression

PC-Find Code

```
PC-Find(x)
    if(x!=Parent(x)){
        Parent(x) = PC-Find(Parent(x));
    }
    return Parent(x);
```

Example

Path compression during Find($c$). Shaded nodes have a new parent.

Rank

- For ease of analysis, instead of keeping track of the size of each of the trees, we will keep track of the rank
- Each node will have an associated rank
- This rank will give an estimate of the log of the number of elements in the set
Code

PC-MakeSet(x){
    parent(x) = x;
    rank(x) = 0;
}

PC-Union(x,y){
    xRep = PC-Find(x);
    yRep = PC-Find(y);
    if(rank(xRep) > rank(yRep))
        parent(yRep) = xRep;
    else{
        parent(xRep) = yRep;
        if(rank(xRep)==rank(yRep))
            rank(yRep)++;
    }
}

Rank Facts

- If an object $x$ is not the set leader, then the rank of $x$ is
  strictly less than the rank of its parent
- For a set $X$, $\text{size}(X) \geq 2^{\text{rank(leader}(X))}$ (can show using induction)
- Since there are $n$ objects, the highest possible rank is $O(\log n)$
- Only set leaders can change their rank

Blocks

- We will also partition the objects into several numbered blocks
- $x$ is assigned to block number $\log^*(\text{rank}(x))$
- Intuitively, $\log^* n$ is the number of times you need to hit the log button on your calculator, after entering $n$, before you get 1
- In other words $x$ is in block $b$ if
  \[ 2 \uparrow\uparrow (b-1) < \text{rank}(x) \leq 2 \uparrow\uparrow b, \]
  where $\uparrow\uparrow$ is defined as in the next slide

Can also say that there are at most $n/2^r$ objects with rank $r$.

- When the rank of a set leader $x$ changes from $r-1$ to $r$, mark all nodes in that set. At least $2^r$ nodes are marked and each of these marked nodes will always have rank less than $r$
- There are $n$ nodes total and any object with rank $r$ marks $2^r$ of them
- Thus there can be at most $n/2^r$ objects of rank $r$
Definition

- $2 \uparrow \uparrow b$ is the tower function
  
  $$2 \uparrow \uparrow b = 2^{2^{\cdots^{2}} b} = \begin{cases} 1 & \text{if } b = 0 \\ 2^{2 \uparrow \uparrow (b-1)} & \text{if } b > 0 \end{cases}$$

Number of Blocks

- Every object has a rank between 0 and $\lfloor \log n \rfloor$
- So the blocks numbers range from 0 to $\log^* \lfloor \log n \rfloor = \log^*(n) - 1$
- Hence there are $\log^* n$ blocks

Number Objects in Block $b$

- Since there are at most $n/2^r$ objects with any rank $r$, the total number of objects in block $b$ is at most
  
  $$\sum_{r=2 \uparrow \uparrow (b-1)+1}^{2 \uparrow \uparrow b} \frac{n}{2^r} < \sum_{r=2 \uparrow \uparrow (b-1)+1}^{\infty} \frac{n}{2^r} = \frac{n}{2^{2 \uparrow \uparrow (b-1)}} = \frac{n}{2 \uparrow \uparrow b}.$$ 

Theorem

- **Theorem:** If we use both PC-Find and PC-Union (i.e. Path Compression and Weighted Union), the worst-case running time of a sequence of $m$ operations, $n$ of which are MakeSet operations, is $O(m \log^* n)$
- Each PC-MakeSet and PC-Union operation takes constant time, so we need only show that any sequence of $m$ PC-Find operations require $O(m \log^* n)$ time in the worst case
- We will use a kind of accounting method to show this
**Proof**

- The cost of PC-Find($x_0$) is proportional to the number of nodes on the path from $x_0$ up to its leader.
- Each object $x_0, x_1, x_2, \ldots, x_l$ on the path from $x_0$ to its leader will pay a $1$ tax into one of several bank accounts.
- After all the Find operations are done, the total amount of money in these accounts will give us the total running time.

**Taxation**

- The leader $x_l$ pays into the leader account.
- The child of the leader $x_{l-1}$ pays into the child account.
- Any other object $x_i$ in a different block from its parent $x_{i+1}$ pays into the block account.
- Any other object $x_i$ in the same block as its parent $x_{i+1}$ pays into the path account.

**Example**

Different nodes on the find path pay into different accounts: B=block, P=path, C=child, L=leader.
Horizontal lines are boundaries between blocks. Only the nodes on the find path are shown.

**Leader, Child and Block accounts**

- During any Find operation, one dollar is paid into the leader account.
- At most one dollar is paid into the child account.
- At most one dollar is paid into the block account for each of the log*$n$ blocks.
- Thus when the sequence of $m$ operations ends, these accounts share a total of at most $2m + m\log^*n$ dollars.
Path Account

- The only remaining difficulty is the Path account
- Consider an object $x_i$ in block $b$ that pays into the path account
- This object is not a set leader so its rank can never change.
- The parent of $x_i$ is also not a set leader, so after path compression, $x_i$ gets a new parent, $x_l$, whose rank is strictly larger than its old parent $x_{i+1}$
- Since $\text{rank}(\text{parent}(x_i))$ is always increasing, parent of $x_i$ must eventually be in a different block than $x_i$, after which $x_i$ will never pay into the path account
- Thus $x_i$ pays into the path account at most once for every rank in block $b$, or less than $2 \uparrow \uparrow b$ times total

Take Away

- We can now say that each call to PC-Find has amortized cost $O(\log^* n)$, which is significantly better than the worst case cost of $O(\log n)$
- The book shows that PC-Find has amortized cost of $O(A(n))$ where $A(n)$ is an even slower growing function than $\log^* n$
- Since block $b$ contains less than $n/(2 \uparrow \uparrow b)$ objects, and each of these objects contributes less than $2 \uparrow \uparrow b$ dollars, the total number of dollars contributed by objects in block $b$ is less than $n$ dollars to the path account
- There are $\log^* n$ blocks so the path account receives less than $n \log^* n$ dollars total
- Thus the total amount of money in all four accounts is less than $2m + m \log^* n + n \log^* n = O(m \log^* n)$, and this bounds the total running time of the $m$ operations.