Today's Outline

- Annihilators for recurrences with non-homogeneous terms
- Transformations

Annihilator Method

- Write down the annihilator for the recurrence
- Factor the annihilator
- Look up the factored annihilator in the “Lookup Table” to get general solution
- Solve for constants of the general solution by using initial conditions

Lookup Table

$$(L - a_0)^{b_0}(L - a_1)^{b_1} \ldots (L - a_k)^{b_k}$$

annihilates only sequences of the form:

$$\langle p_1(n)a_0^n + p_2(n)a_1^n + \ldots + p_k(n)a_k^n \rangle$$

where $p_i(n)$ is a polynomial of degree $b_i - 1$ (and $a_i \neq a_j$, when $i \neq j$)
Examples

- Q: What does \((L - 3)(L - 2)(L - 1)\) annihilate?
  - A: \(c_01^n + c_12^n + c_23^n\)
- Q: What does \((L - 3)^2(L - 2)(L - 1)\) annihilate?
  - A: \(c_01^n + c_12^n + (c_2n + c_3)3^n\)
- Q: What does \((L - 1)^4\) annihilate?
  - A: \((c_0n^3 + c_1n^2 + c_2n + c_3)1^n\)
- Q: What does \((L - 1)^3(L - 2)^2\) annihilate?
  - A: \((c_0n^2 + c_1n + c_2)1^n + (c_3n + c_4)2^n\)

Example

Consider the recurrence \(T(n) = 7T(n - 1) - 16T(n - 2) + 12T(n - 3)\), \(T(0) = 1\), \(T(1) = 5\), \(T(2) = 17\)

- Write down the annihilator: From the definition of the sequence, we can see that \(L^3T - 7L^2T + 16LT - 12T = 0\), so the annihilator is \(L^3 - 7L^2 + 16L - 12\)
- Factor the annihilator: We can factor by hand or using a computer program to get \(L^3 - 7L^2 + 16L - 12 = (L - 2)(L - 3)(L - 4)\)
- Look up to get general solution: The annihilator \((L - 2)^2(L - 3)\) annihilates sequences of the form \(\langle c_0n + c_1\rangle^n + c_23^n\)
- Solve for constants: \(T(0) = 1 = c_1 + 2, T(1) = 5 = 2c_0 + 2c_1 + 3c_2, T(2) = 17 = 8c_0 + 4c_1 + 9c_2\). We’ve got three equations and three unknowns. Solving by hand, we get that \(c_0 = 1, c_1 = 0, c_2 = 1\). Thus: \(T(n) = n2^n + 3^n\)

Example (II)

Consider the recurrence \(T(n) = 2T(n - 1) - T(n - 2)\), \(T(0) = 0\), \(T(1) = 1\)

- Write down the annihilator: From the definition of the sequence, we can see that \(L^2T - 2LT + T = 0\), so the annihilator is \(L^2 - 2L + 1\)
- Factor the annihilator: We can factor by hand or using the quadratic formula to get \(L^2 - 2L + 1 = (L - 1)^2\)
- Look up to get general solution: The annihilator \((L - 1)^2\) annihilates sequences of the form \(\langle c_0n + c_1\rangle^n\)
- Solve for constants: \(T(0) = 0 = c_1, T(1) = 1 = c_0 + c_1\). We’ve got two equations and two unknowns. Solving by hand, we get that \(c_0 = 0, c_1 = 1\). Thus: \(T(n) = n\)

At Home Exercise

Consider the recurrence \(T(n) = 6T(n - 1) - 9T(n - 2)\), \(T(0) = 1\), \(T(1) = 6\)

- Q1: What is the annihilator of this sequence?
- Q2: What is the factored version of the annihilator?
- Q3: What is the general solution for the recurrence?
- Q4: What are the constants in this general solution?

(Note: You can check that your general solution works for \(T(2)\))
Non-homogeneous terms

- Consider a recurrence of the form $T(n) = T(n - 1) + T(n - 2) + k$ where $k$ is some constant
- The terms in the equation involving $T$ (i.e. $T(n - 1)$ and $T(n - 2)$) are called the homogeneous terms
- The other terms (i.e. $k$) are called the non-homogeneous terms

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Example

- In a *height-balanced tree*, the height of two subtrees of any node differ by at most one
- Let $T(n)$ be the smallest number of nodes needed to obtain a height balanced binary tree of height $n$
- Q: What is a recurrence for $T(n)$?
- A: Divide this into smaller subproblems
  - To get a height-balanced tree of height $n$ with the smallest number of nodes, need one subtree of height $n - 1$, and one of height $n - 2$, plus a root node
  - Thus $T(n) = T(n - 1) + T(n - 2) + 1$

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Example

- Let's solve this recurrence: $T(n) = T(n - 1) + T(n - 2) + 1$
  (Let $T_n = T(n)$, and $T = \langle T_n \rangle$)
- We know that $(L^2 - L - 1)$ annihilates the homogeneous terms
- Let's apply it to the entire equation:
  $$
  (L^2 - L - 1)\langle T_n \rangle = L^2\langle T_n \rangle - L\langle T_n \rangle - 1\langle T_n \rangle
  = \langle T_{n+2} \rangle - \langle T_{n+1} \rangle - \langle T_n \rangle
  = \langle T_{n+2} - T_{n+1} - T_n \rangle
  = \langle 1, 1, 1, \cdots \rangle
  $$

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Example

- This is close to what we want but we still need to annihilate $\langle 1, 1, 1, \cdots \rangle$
- It's easy to see that $L - 1$ annihilates $\langle 1, 1, 1, \cdots \rangle$
- Thus $(L^2 - L - 1)(L - 1)$ annihilates $T(n) = T(n - 1) + T(n - 2) + 1$
- When we factor, we get $(L - \phi)(L - \tilde{\phi})(L - 1)$, where $\phi = \frac{1 + \sqrt{5}}{2}$ and $\tilde{\phi} = \frac{1 - \sqrt{5}}{2}$.
Lookup

• Looking up \((L - \phi)(L - \hat{\phi})(L - 1)\) in the table
• We get \(T(n) = c_1\phi^n + c_2\hat{\phi}^n + c_31^n\)
• If we plug in the appropriate initial conditions, we can solve for these three constants
• We’ll need to get equations for \(T(2)\) in addition to \(T(0)\) and \(T(1)\)

General Rule

To find the annihilator for recurrences with non-homogeneous terms, do the following:

• Find the annihilator \(a_1\) for the homogeneous part
• Find the annihilator \(a_2\) for the non-homogeneous part
• The annihilator for the whole recurrence is then \(a_1a_2\)

Another Example

• Consider \(T(n) = T(n - 1) + T(n - 2) + 2\).
• The residue is \(\langle 2, 2, 2, \cdots \rangle\) and
• The annihilator is still \((L^2 - L - 1)(L - 1)\), but the equation for \(T(2)\) changes!

Another Example

• Consider \(T(n) = T(n - 1) + T(n - 2) + 2n\).
• The residue is \(\langle 1, 2, 4, 8, \cdots \rangle\) and
• The annihilator is now \((L^2 - L - 1)(L - 2)\).
Another Example

• Consider \( T(n) = T(n - 1) + T(n - 2) + n \).
• The residue is \( \langle 1, 2, 3, 4, \ldots \rangle \).
• The annihilator is now \( (L^2 - L - 1)(L - 1)^2 \).

Another Example

• Consider \( T(n) = T(n - 1) + T(n - 2) + n^2 \).
• The residue is \( \langle 1, 4, 9, 16, \ldots \rangle \) and
• The annihilator is \( (L^2 - L - 1)(L - 1)^3 \).

Another Example

• Consider \( T(n) = T(n - 1) + T(n - 2) + n^2 + 2n \).
• The residue is \( \langle 1 + 1, 4 + 4, 9 + 8, 16 + 16, \ldots \rangle \) and the
• The annihilator is \( (L^2 - L - 1)(L - 1)^3(L - 2) \).

In Class Exercise

• Consider \( T(n) = 3 \ast T(n - 1) + 3^n \)
• Q1: What is the homogeneous part, and what annihilates it?
• Q2: What is the non-homogeneous part, and what annihilates it?
• Q3: What is the annihilator of \( T(n) \), and what is the general form of the recurrence?
Limitations

- Our method does not work on $T(n) = T(n-1) + \frac{1}{n}$ or $T(n) = T(n-1) + \log n$.
- The problem is that $\frac{1}{n}$ and $\log n$ do not have annihilators.
- Our tool, as it stands, is limited.
- Key idea for strengthening it is transformations.

Transformations Idea

- Consider the recurrence giving the run time of mergesort $T(n) = 2T(n/2) + kn$ (for some constant $k$), $T(1) = 1$.
- How do we solve this?
- We have no technique for annihilating terms like $T(n/2)$.
- However, we can transform the recurrence into one with which we can work.

Transformation

- Let $n = 2^i$ and rewrite $T(n)$:
  - $T(2^0) = 1$ and $T(2^i) = 2T(\frac{2^i}{2}) + k2^i = 2T(2^{i-1}) + k2^i$.
- Now define a new sequence $t$ as follows: $t(i) = T(2^i)$.
- Then $t(0) = 1$, $t(i) = 2t(i-1) + k2^i$.

Now Solve

- We’ve got a new recurrence: $t(0) = 1$, $t(i) = 2t(i-1) + k2^i$.
- We can easily find the annihilator for this recurrence:
  - $(L-2)$ annihilates the homogeneous part, $(L-2)$ annihilates the non-homogeneous part, so $(L-2)(L-2)$ annihilates $t(i)$.
- Thus $t(i) = (c_1i + c_2)2^i$. 
Reverse Transformation

- We’ve got a solution for \( t(i) \) and we want to transform this into a solution for \( T(n) \).
- Recall that \( t(i) = T(2^i) \) and \( 2^i = n \).

\[
\begin{align*}
  t(i) & = (c_1 i + c_2)2^i & (1) \\
  T(2^i) & = (c_1 i + c_2)2^i & (2) \\
  T(n) & = (c_1 \lg n + c_2)n & (3) \\
  & = c_1 \lg n + c_2 n & (4) \\
  & = \Theta(n \lg n) & (5)
\end{align*}
\]

Success!

Let’s recap what just happened:

- We could not find the annihilator of \( T(n) \) so:
  - We did a transformation to a recurrence we could solve, \( t(i) \) (we let \( n = 2^i \) and \( t(i) = T(2^i) \)).
  - We found the annihilator for \( t(i) \), and solved the recurrence for \( t(i) \).
  - We reverse transformed the solution for \( t(i) \) back to a solution for \( T(n) \).

Another Example

- Consider the recurrence \( T(n) = 9T(\frac{n}{3}) + kn \), where \( T(1) = 1 \) and \( k \) is some constant.
- Let \( n = 3^i \) and rewrite \( T(n) \):
  - \( T(2^0) = 1 \) and \( T(3^i) = 9T(3^{i-1}) + k3^i \)
  - Now define a sequence \( t \) as follows \( t(i) = T(3^i) \)
  - Then \( t(0) = 1 \), \( t(i) = 9t(i - 1) + k3^i \)

Now Solve

- \( t(0) = 1 \), \( t(i) = 9t(i - 1) + k3^i \)
  - This is annihilated by \((L - 9)(L - 3)\)
  - So \( t(i) \) is of the form \( t(i) = c_1 9^i + c_2 3^i \)
### Reverse Transformation

- \( t(i) = c_19^i + c_23^i \)
- Recall: \( t(i) = T(3^i) \) and \( 3^i = n \)
  
  \[
  \begin{align*}
  t(i) &= c_19^i + c_23^i \\
  T(3^i) &= c_19^i + c_23^i \\
  T(n) &= c_1(3^i)^2 + c_23^i \\
  &= c_1n^2 + c_2n \\
  &= \Theta(n^2)
  \end{align*}
  \]

### In Class Exercise

Consider the recurrence \( T(n) = 2T(n/4) + kn \), where \( T(1) = 1 \), and \( k \) is some constant

- Q1: What is the transformed recurrence \( t(i) \)? How do we rewrite \( n \) and \( T(n) \) to get this sequence?
- Q2: What is the annihilator of \( t(i) \)? What is the solution for the recurrence \( t(i) \)?
- Q3: What is the solution for \( T(n) \)? (i.e. do the reverse transformation)

### A Final Example

Not always obvious what sort of transformation to do:

- Consider \( T(n) = 2T(\sqrt{n}) + \log n \)
- Let \( n = 2^i \) and rewrite \( T(n) \):
  
  \[
  \begin{align*}
  T(2^i) &= 2T(2^{i/2}) + i \\
  \text{Define } t(i) &= T(2^i): \\
  t(i) &= 2t(i/2) + i
  \end{align*}
  \]

- This final recurrence is something we know how to solve!
  
  - \( t(i) = O(i \log i) \)
  - The reverse transform gives:
    
    \[
    \begin{align*}
    t(i) &= O(i \log i) & (6) \\
    T(2^i) &= O(i \log i) & (7) \\
    T(n) &= O(\log n \log \log n) & (8)
    \end{align*}
    \]
Todo

- HW 1
- Start Chapter 15 in text