CS 362, HW1

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Due: Feb. 4th

Exercise number are all from the third edition of Cormen, Leiserson, Rivest and Stein. Remember: you are encouraged to work on the homework in groups, but please observe the "Star Trek" rule from the syllabus.

1. Exercise 3.1-5 (a single exercise): Prove Theorem 3.1

2. Problem 3-2 (Relative asymptotic growths)

3. Prove that $\log n! = \Theta(n \log n)$ and that $n! = \omega(2^n)$ and $n! = o(n^n)$

4. Assume you have functions f and g, such that $f(n)$ is $O(g(n))$. For each of the following statements, decide whether you think it is true or false and give either a proof or a counterexample

   (a) $\log_2 f(n)$ is $O(\log_2 g(n))$
   (b) $2^{f(n)}$ is $O(2^{g(n)})$
   (c) $f(n)^2$ is $O(g(n)^2)$

5. A c-tree is a tree with each node colored either red, green or silver that obeys the following rules:

   - Each red node has two children, exactly one of which is green.
   - Each green node has exactly one child, which is not green
   - Silver nodes have no children.

   Let R and S respectively denote the number of red and silver nodes, and n be the total number of nodes. Prove by induction that in any c-tree with $n \geq 1$, $S = R + 1$.

6. Imagine you are doing a stress test on a particular model of smart phone. You have a ladder with n rungs. You want to determine

the highest *safe* step: the highest step from which the phone can be dropped without breaking. You want to do this with the smallest number of phone drops.

(a) Imagine that you have exactly 2 phones. Devise an algorithm that can determine the highest safe step using $o(n)$ drops.

(b) Now suppose you have k phones. Devise an algorithm that can determine the highest safe step with the smallest number of drops. If $f(k, n)$ is the number of drops that your algorithm needs, what is $f(k, n)$ asymptotically? Hint: you should ensure that $f(k, n) = o(f(k-1, n))$ for any constant k.

7. The game of Match is played with a special deck of 27 cards. Each card has three attributes: color, shape and number. The possible color values are {red, blue, green}, the possible shape values are {square, circle, heart}, and the possible number values are {1, 2, 3}. Each of the $3 * 3 * 3 = 27$ possible combinations is represented by a card in the deck. A match is a set of 3 cards with the property that for every one of the three attributes, either all the cards have the same value for that attribute or they all have different values for that attribute. For example, the following three cards are a match: (3, red, square), (2, blue, square), (1, green, square).

   (a) If we shuffle the deck and turn over three cards, what is the probability that they form a match? Hint: given the first two cards, what is the probability that the third forms a match?

   (b) If we shuffle the deck and turn over n cards where $n \leq 27$, what is the expected number of matches, where we count each match separately even if they overlap? Note: The cards in a match do not need to be adjacent! Is your expression correct for $n = 27$?

8. Drunken Debutants: Imagine that there are n debutants, each with her own Porsche. After a late and wild party, each debutante stumbles into a Porsche selected independently and uniformly at random (thus, more than one debutant may wind up in a given Porsche). Let X be a random variable giving the number of debutants that wind up in their own Porsche. Use linearity of expectation to compute the expected value of X. Now use Markov's inequality, to bound the probability that X is larger than k for any positive k.

9. Imagine n points are distributed independently and uniformly at random on the circumference of a circle that has circumference of length 1. Let the distance between a pair of points on the circumference be the length of the arc between them. Show that the expected number of pairs of points that are within distance $\Theta(1/n^2)$ of each other is greater than 1. FYI: this problem has applications in efficient routing in peer-to-peer networks.

   Hint: Partition the circumference of the circle into n^2/k arcs of length k/n^2 for some constant k; then use the Birthday paradox to solve for the necessary k.