CS 362, All Pairs Shortest Paths

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Outline

• All Pairs Shortest Paths
• Floyd Warshall Algorithm
All-Pairs Shortest Paths

- For the single-source shortest paths problem, we wanted to find the shortest path from a source vertex $s$ to all the other vertices in the graph.
- We will now generalize this problem further to that of finding the shortest path from every possible source to every possible destination.
- In particular, for every pair of vertices $u$ and $v$, we need to compute the following information:
  - $dist(u, v)$ is the length of the shortest path (if any) from $u$ to $v$.
  - $pred(u, v)$ is the second-to-last vertex (if any) on the shortest path (if any) from $u$ to $v$. 

Example

- For any vertex $v$, we have $\text{dist}(v, v) = 0$ and $\text{pred}(v, v) = \text{NULL}$
- If the shortest path from $u$ to $v$ is only one edge long, then $\text{dist}(u, v) = w(u \rightarrow v)$ and $\text{pred}(u, v) = u$
- If there’s no shortest path from $u$ to $v$, then $\text{dist}(u, v) = \infty$ and $\text{pred}(u, v) = \text{NULL}$
The output of our shortest path algorithm will be a pair of $|V| \times |V|$ arrays encoding all $|V|^2$ distances and predecessors. Many maps contain such a distance matrix - to find the distance from (say) Albuquerque to (say) Ruidoso, you look in the row labeled “Albuquerque” and the column labeled “Ruidoso”.

In this class, we’ll focus only on computing the distance array.

The predecessor array, from which you would compute the actual shortest paths, can be computed with only minor additions to the algorithms presented here.
Lots of Single Sources

- Most obvious solution to APSP is to just run SSSP algorithm $|V|$ times, once for every possible source vertex.
- Specifically, to fill in the subarray $dist(s, \ast)$, we invoke either Dijkstra’s or Bellman-Ford starting at the source vertex $s$.
- We’ll call this algorithm ObviousAPSP.
ObviousAPSP(V,E,w){
    for every vertex s{
        dist(s,*) = SSSP(V,E,w,s);
    }
}
The running time of this algorithm depends on which SSSP algorithm we use.

- If we use Bellman-Ford, the overall running time is $O(|V|^2|E|) = O(|V|^4)$.
- If all the edge weights are positive, we can use Dijkstra’s instead, which decreases the run time to $O(|V||E| + |V|^2 \log |V|) = O(|V|^3)$. 
Problem

- We’d like to have an algorithm which takes $O(|V|^3)$ but which can also handle negative edge weights
- We’ll see that a dynamic programming algorithm, the Floyd Warshall algorithm, will achieve this
- Note: the book discusses another algorithm, Johnson’s algorithm, which is asymptotically better than Floyd Warshall on sparse graphs. However we will not be discussing this algorithm in class.
Dynamic Programming

- Recall: Dynamic Programming = Recursion + Memorization
- Thus we first need to come up with a recursive formulation of the problem
- We might recursively define $dist(u, v)$ as follows:

$$dist(u, v) = \begin{cases} 
0 & \text{if } u = v \\
\min_x (dist(u, x) + w(x \rightarrow v)) & \text{otherwise}
\end{cases}$$
The problem

- In other words, to find the shortest path from $u$ to $v$, try all possible predecessors $x$, compute the shortest path from $u$ to $x$ and then add the last edge $u \rightarrow v$
- **Unfortunately, this recurrence doesn’t work**
  - To compute $\text{dist}(u, v)$, we first must compute $\text{dist}(u, x)$ for every other vertex $x$, but to compute any $\text{dist}(u, x)$, we first need to compute $\text{dist}(u, v)$
  - We’re stuck in an infinite loop!
The solution

- To avoid this circular dependency, we need some additional parameter that decreases at each recursion and eventually reaches zero at the base case
- One possibility is to include the number of edges in the shortest path as this third magic parameter
- So define $\text{dist}(u, v, k)$ to be the length of the shortest path from $u$ to $v$ that uses at most $k$ edges
- Since we know that the shortest path between any two vertices uses at most $|V| - 1$ edges, what we want to compute is $\text{dist}(u, v, |V| - 1)$
The Recurrence

\[ \text{dist}(u, v, k) = \begin{cases} 
0 & \text{if } u = v \\
\infty & \text{if } k = 0 \text{ and } u \neq v \\
\min_x \left( \text{dist}(u, x, k - 1) + w(x \to v) \right) & \text{otherwise}
\end{cases} \]
The Algorithm

- It's not hard to turn this recurrence into a dynamic programming algorithm
- Even before we write down the algorithm, though, we can tell that its running time will be $\Theta(|V|^4)$
- This is just because the recurrence has four variables — $u$, $v$, $k$ and $x$ — each of which can take on $|V|$ different values
- Except for the base cases, the algorithm will just be four nested “for” loops
DP-APSP

DP-APSP(V,E,w){
    for all vertices u in V{
        for all vertices v in V{
            if(u=v)
                dist(u,v,0) = 0;
            else
                dist(u,v,0) = infinity;
        }
    }
    for k=1 to |V|-1{
        for all vertices u in V{
            for all vertices u in V{
                dist(u,v,k) = infinity;
                for all vertices x in V{
                    if (dist(u,v,k)>dist(u,x,k-1)+w(x,v))
                        dist(u,v,k) = dist(u,x,k-1)+w(x,v);
                }
            }
        }
    }}}}}}
The Problem

- This algorithm still takes $O(|V|^4)$ which is no better than the ObviousAPSP algorithm
- If we use a certain divide and conquer technique, there is a way to get this down to $O(|V|^3 \log |V|)$ (think about how you might do this)
- However, to get down to $O(|V|^3)$ run time, we need to use a different third parameter in the recurrence
Floyd-Warshall

- Number the vertices arbitrarily from 1 to $|V|$
- Define $dist(u, v, r)$ to be the shortest path from $u$ to $v$ where all intermediate vertices (if any) are numbered $r$ or less
- If $r = 0$, we can’t use any intermediate vertices so shortest path from $u$ to $v$ is just the weight of the edge (if any) between $u$ and $v$
- If $r > 0$, then either the shortest legal path from $u$ to $v$ goes through vertex $r$ or it doesn’t
- We need to compute the shortest path distance from $u$ to $v$ with no restrictions, which is just $dist(u, v, |V|)$
The recurrence

We get the following recurrence:

\[
\text{dist}(u, v, r) = \begin{cases} 
  w(u \rightarrow v) & \text{if } r = 0 \\
  \min\{\text{dist}(u, v, r - 1), \\
  \quad \text{dist}(u, r, r - 1) + \text{dist}(r, v, r - 1)\} & \text{otherwise}
\end{cases}
\]
The Algorithm

FloydWarshall(V,E,w){
    for u=1 to |V|{
        for v=1 to |V|{
            dist(u,v,0) = w(u,v);
        }}
    for r=1 to |V|{
        for u=1 to |V|{
            for v=1 to |V|{
                if (dist(u,v,r-1) < dist(u,r,r-1) + dist(r,v,r-1))
                    dist(u,v,r) = dist(u,v,r-1);
                else
                    dist(u,v,r) = dist(u,r,r-1) + dist(r,v,r-1);
            }}}}
• There are three variables here, each of which takes on $|V|$ possible values
• Thus the run time is $\Theta(|V|^3)$
• Space required is also $\Theta(|V|^3)$
Take Away

- Floyd-Warshall solves the APSP problem in $\Theta(|V|^3)$ time even with negative edge weights
- Floyd-Warshall uses dynamic programming to compute APSP
- We’ve seen that sometimes for a dynamic program, we need to introduce an *extra variable* to break dependencies in the recurrence.
- We’ve also seen that the choice of this extra variable can have a big impact on the run time of the dynamic program