CS 362, Randomized Data Structures: Hash Tables, Skip Lists, Bloom Filters, Count-Min sketch

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Outline

- Skip Lists
- Count-Min Sketch
A dictionary ADT implements the following operations

- \textit{Insert}(x): puts the item \( x \) into the dictionary
- \textit{Delete}(x): deletes the item \( x \) from the dictionary
- \textit{IsIn}(x): returns true iff the item \( x \) is in the dictionary
Dictionary ADT

- Frequently, we think of the items being stored in the dictionary as *keys*.
- The keys typically have *records* associated with them which are carried around with the key but not used by the ADT implementation.
- Thus we can implement functions like:
  - $\text{Insert}(k, r)$: puts the item $(k, r)$ into the dictionary if the key $k$ is not already there, otherwise returns an error.
  - $\text{Delete}(k)$: deletes the item with key $k$ from the dictionary.
  - $\text{Lookup}(k)$: returns the item $(k, r)$ if $k$ is in the dictionary, otherwise returns null.
Hash Tables

Hash Tables implement the Dictionary ADT, namely:

- Insert(x) - $O(1)$ expected time, $\Theta(n)$ worst case
- Lookup(x) - $O(1)$ expected time, $\Theta(n)$ worst case
- Delete(x) - $O(1)$ expected time, $\Theta(n)$ worst case

Question: Can we do better? Yes with skip lists!
Skip List

- Enables insertions and searches for ordered keys in $O(\log n)$ expected time
- Very elegant randomized data structure, simple to code but analysis is subtle
- They guarantee that, with high probability, all the major operations take $O(\log n)$ time (e.g. Search, Insert, Delete, Find-Max, Predecessor/ Sucessor)
- Can even enable ”find-i-th value” if store with each edge the number of elements that edge skips
Skip List

- A skip list is basically a collection of doubly-linked lists, $L_1, L_2, \ldots, L_x$, for some integer $x$
- Each list has a special head and tail node, the keys of these nodes are assumed to be $-\infty$ and $+\infty$ respectively
- The keys in each list are in sorted order (non-decreasing)
Skip List

- Nodes are added by calls to Insert() (next slides)
- Intuitively, every time we insert an item, we’ll always insert it into the bottom list, and insert into lists above that probabilistically (for details, Insert() code below)
- Each node $v$ stores a search key (key($v$)), a pointer to its next lower copy (down($v$)), and a pointer to the next node in its level (right($v$)).
Example
Search

- To do a search for a key, $k$, we start at the leftmost node $L$ in the highest level.
- We then scan through each level as far as we can without passing the target value $x$ and then proceed down to the next level.
- The search ends either when we find the key $x$ or fail to find $x$ on the lowest level.
Search(k)

LookUp(k, L){
    v = L;
    while (v != NULL) and (Key(v) != k){
        if (Key(Right(v)) > k)
            v = Down(v);
        else
            v = Right(v);
    }
    return v;
}

Search Example
Insert(k)

coin() returns "heads" with probability 1/2

Insert(k){
First call Search(k), let pLeft be the leftmost elem <= k in L_1
Insert k in List 0, to the right of pLeft
i = 1;
while (coin() = "heads"){
   insert k in List i;
   i++;
}

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Deletion

- Deletion is very simple
- First do a search for the key to be deleted
- Then delete that key from all the lists it appears in from the bottom up, making sure to “zip up” the lists after the deletion
Analysis

• Intuitively, each level of the skip list has about half the number of nodes of the previous level, so we expect the total number of levels to be about $O(\log n)$
• Similarly, each time we add another level, we cut the search time in half except for a constant overhead
• So after $O(\log n)$ levels, we would expect a search time of $O(\log n)$
• We will now formalize these two intuitive observations
Height of Skip List

- For some key, $k$, let $X_k$ be the maximum height of $k$ in the skip list.
- Q: What is the probability that $X_k \geq 2\log n$?
- A: If $p = 1/2$, we have:

\[
P(X_k \geq 2\log n) = \left(\frac{1}{2}\right)^{2\log n} = \frac{1}{(2^{\log n})^2} = \frac{1}{n^2}
\]

- Thus the probability that a particular key $k$ achieves height $2\log n$ is $\frac{1}{n^2}$
Height of Skip List

- Q: What is the probability that any key achieves height $2 \log n$?
- A: We want

$$P(X_1 \geq 2 \log n \text{ or } X_2 \geq 2 \log n \text{ or } \ldots \text{ or } X_n \geq 2 \log n)$$

- By a Union Bound, this probability is no more than

$$P(X_1 \geq 2 \log n) + P(X_2 \geq 2 \log n) + \cdots + P(X_n \geq 2 \log n)$$

- Which equals:

$$\sum_{i=1}^{n} \frac{1}{n^2} = \frac{n}{n^2} = 1/n$$
Height of Skip List

- This probability gets small as $n$ gets large
- In particular, the probability of having a skip list of height exceeding $2 \log n$ is $o(1)$
- If an event occurs with probability $1 - o(1)$, we say that it occurs with high probability
- Key Point: The height of a skip list is $O(\log n)$ with high probability.
In-Class Exercise Trick

A trick for computing expectations of discrete positive random variables:

- Let $X$ be a discrete r.v., that takes on values from 1 to $n$

\[
E(X) = \sum_{i=1}^{n} P(X \geq i)
\]
Why?

\[
\sum_{i=1}^{n} P(X \geq i) = P(X = 1) + P(X = 2) + P(X = 3) + \ldots \\
+ P(X = 2) + P(X = 3) + P(X = 4) + \ldots \\
+ P(X = 3) + P(X = 4) + P(X = 5) + \ldots \\
+ \ldots \\
= 1 \times P(X = 1) + 2 \times P(X = 2) + 3 \times P(X = 3) + \ldots \\
= E(X)
\]
In-Class Exercise

Q: How much memory do we expect a skip list to use up?

• Let $X_k$ be the number of lists that key $k$ is inserted in.
• Q: What is $P(X_k \geq 1)$, $P(X_k \geq 2)$, $P(X_k \geq 3)$?
• Q: What is $P(X_k \geq i)$ for $i \geq 1$?
• Q: What is $E(X_k)$?
• Q: Let $X = \sum_{k=1}^{n} X_k$. What is $E(X)$?
Search Time

• It’s easier to analyze the search time if we imagine running the search backwards
• Imagine that we start at the found node $v$ in the bottommost list and we trace the path backwards to the top leftmost sentinel, $L$
• This will give us the length of the search path from $L$ to $v$ which is the time required to do the search
Backwards Search

SLFback(v) {
    while (v != L) {
        if (Up(v) != NIL)
            v = Up(v);
        else
            v = Left(v);
    }
}
Backward Search

- For every node \( v \) in the skip list \( \text{Up}(v) \) exists with probability \( 1/2 \). So for purposes of analysis, SLFBack is the same as the following algorithm:

```java
FlipWalk(v){
    while (v != L){
        if (COINFLIP == HEADS)
            v = Up(v);
        else
            v = Left(v);
    }
}
```
Analysis

- For this algorithm, the expected number of heads is exactly the same as the expected number of tails.
- Thus the expected run time of the algorithm is twice the expected number of upward jumps.
- Since we already know that the number of upward jumps is $O(\log n)$ with high probability, we can conclude that the expected search time is $O(\log n)$. 
Bloom Filters

- Randomized data structure for representing a set. Implemen-ments:
- Insert(x):
- IsMember(x):
- Allow false positives but require very little space
- Used frequently in: Databases, networking problems, p2p networks, packet routing
Bloom Filters

- Have $m$ slots, $k$ hash functions, $n$ elements; assume hash functions are all independent
- Each slot stores 1 bit, initially all bits are 0
- Insert($x$) : Set the bit in slots $h_1(x), h_2(x), ..., h_k(x)$ to 1
- IsMember($x$) : Return yes iff the bits in $h_1(x), h_2(x), ..., h_k(x)$ are all 1
Analysis Sketch

- $m$ slots, $k$ hash functions, $n$ elements; assume hash functions are all independent
- Then $P(\text{fixed slot is still 0}) = (1 - 1/m)^{kn}$
- Useful fact from Taylor expansion of $e^{-x}$:
  
  $e^{-x} - x^2/2 \leq 1 - x \leq e^{-x}$ for $x < 1$

- Then if $x \leq 1$

  $$e^{-x}(1 - x^2) \leq 1 - x \leq e^{-x}$$
Analysis

• Thus we have the following to good approximation.

\[ Pr(\text{fixed slot is still 0}) = (1 - 1/m)^{kn} \approx e^{-kn/m} \]

• Let \( p = e^{-kn/m} \) and let \( \rho \) be the fraction of 0 bits after \( n \) elements inserted then

\[ Pr(\text{false positive}) = (1 - \rho)^k \approx (1 - p)^k \]

• Where this last approximation holds because \( \rho \) is very close to \( p \) (by a Martingale argument beyond the scope of this class)
• Want to minimize \((1 - p)^k\), which is equivalent to minimizing 
  \[ g = k \ln(1 - p) \]
• Trick: Note that 
  \[ g = -(m/n) \ln(p) \ln(1 - p) \]
• By symmetry, this is minimized when \(p = 1/2\) or equivalently 
  \[ k = (m/n) \ln 2 \]
• False positive rate is then 
  \[ (1/2)^k \approx (.6185)^{m/n} \]
Tricks

- Can get the union of two sets by just taking the bitwise-or of the bit-vectors for the corresponding Bloom filters
- Can easily half the size of a bloom filter - assume size is power of 2 then just bitwise-or the first and second halves together
- Can approximate the size of the intersection of two sets - inner product of the bit vectors associated with the Bloom filters is a good approximation to this.
Extensions

- Bloomier Filters: Also allow for data to be inserted in the filter - similar functionality to hash tables but less space, and the possibility of false positives.
Data Streams

- A router forwards packets through a network
- A natural question for an administrator to ask is: what is the list of sub-strings of a fixed length that have passed through the router more than a predetermined threshold number of times
- This would be a natural way to try to, for example, identify worms and spam
- Problem: the number of packets passing through the router is *much* too high to be able to store counts for every sub-string that is seen!
Data Streams

- This problem motivates the data stream model
- Informally: there is a stream of data given as input to the algorithm
- The algorithm can take at most one pass over this data and must process it sequentially
- The memory available to the algorithm is much less than the size of the stream
- In general, we won’t be able to solve problems exactly in this model, only approximate
Our Problem

- We are presented with a stream of items \( i \)
- We want to get a good approximation to the value \( \text{Count}(i, T) \), which is the number of times we have seen item \( i \) up to time \( T \)
Count-Min Sketch

- Our solution will be to use a data structure called a *Count-Min Sketch*
- This is a randomized data structure that will keep approximate values of Count(i, T)
- It is implemented using $k$ hash functions and $m$ counters
Count-Min Sketch

- Think of our $m$ counters as being in a 2-dimensional array, with $m/k$ counters per row and $k$ rows
- Let $C_{a,b}$ be the counter in row $a$ and column $b$
- Our hash functions map items from the universe into counters
- In particular, hash function $h_a$ maps item $i$ to counter $C_{a,h_a(i)}$
Updates

- Initially all counters are set to 0
- When we see item \( i \) in the data stream we do the following
- For each \( 1 \leq a \leq k \), increment \( C_{a,h_a(i)} \)
Count Approximations

• Let $C_{a,b}(T)$ be the value of the counter $C_{a,b}$ after processing $T$ tuples
• We approximate $\text{Count}(i,T)$ by returning the value of the smallest counter associated with $i$
• Let $m(i,T)$ be this value
Main Theorem:

- For any item $i$, $m(i, T) \geq \text{Count}(i, T)$
- With probability at least $1 - e^{-m\epsilon/e}$ the following holds: $m(i, T) \leq \text{Count}(i, T) + \epsilon T$
Proof

- Easy to see that $m(i, T) \geq \text{Count}(i, T)$, since each counter $C_{a,h_a(i)}$ incremented by $c_t$ every time pair $(i, c_t)$ is seen.
- Hard Part: Showing $m(i, T) \leq \text{Count}(i, T) + \epsilon T$.
- To see this, we will first consider the specific counter $C_{1,h_1(i)}$ and then use symmetry.
Proof

- Let $Z_1$ be a random variable giving the amount the counter is incremented by items other than $i$
- Let $X_t$ be an indicator r.v. that is 1 if $j$ is the $t$-th item, and $j \neq i$ and $h_1(i) = h_1(j)$
- Then $Z_1 = \sum_{t=1}^{T} X_t$
- But if the hash functions are “good”, then if $i \neq j$, $Pr(h_1(i) = h_1(j)) \leq k/m$ (specifically, we need the hash functions to come from a 2-universal family, but we won’t get into that in this class)
- Hence, $E(X_t) \leq k/m$
Proof

- Thus, by linearity of expectation, we have that:

\[ E(Z_1) = \sum_{t=1}^{T} (k/m) \leq Tk/m \]  

- We now need to make use of a very important inequality: Markov’s inequality
Markov’s Inequality

• Let $X$ be a random variable that only takes on non-negative values
• Then for any $\lambda > 0$:

$$Pr(X \geq \lambda) \leq \frac{E(X)}{\lambda}$$

• Proof of Markov's: Assume instead that there exists a $\lambda$ such that $Pr(X \geq \lambda)$ was actually larger than $E(X)/\lambda$
• But then the expected value of $X$ would be at least $\lambda \cdot Pr(X \geq \lambda) > E(X)$, which is a contradiction!!!
Proof

- Now, by Markov’s inequality,

\[ \Pr(Z_1 \geq \epsilon T') \leq \frac{(Tk/m)/(\epsilon T')}{k/(m\epsilon)} = k/(m\epsilon) \]

- This is the event where \( Z_1 \) is “bad” for item \( i \).
Proof (Cont’d)

• Now again assume our $k$ hash functions are “good” in the sense that they are independent
• Then we have that

$$\prod_{i=1}^{k} \Pr(Z_j \geq \epsilon T) \leq \left(\frac{k}{m \epsilon}\right)^k$$
Proof

- Finally, we want to choose a $k$ that minimizes $f(k) = \left(\frac{k}{m\epsilon}\right)^k$
- Note that $\frac{\partial f}{\partial k} = \left(\frac{k}{m\epsilon}\right)^k \left(\ln \frac{k}{m\epsilon} + 1\right)$
- From this, we can see that the probability is minimized when $k = m\epsilon/e$, in which case

$$\left(\frac{k}{m\epsilon}\right)^k = e^{-m\epsilon/e}$$

- This completes the proof!
Recap

- Our Count-Min Sketch is very good at giving estimating counts of items with very little external space.
- Tradeoff is that it only provides approximate counts, but we can bound the approximation!
- Note: Can use the Count-Min Sketch to keep track of all the items in the stream that occur more than a given threshold ("heavy hitters")
- Basic idea is to store an item in a list of "heavy hitters" if its count estimate ever exceeds some given threshold