1. Your boss asks you to decide which of two algorithms to use in a new software system. The runtimes of the two algorithms are given by the following recurrences (remember that when the base case of a recurrence is not given, assume $T(c) = \Theta(1)$ for any constant $c$):

   - Algorithm 1: $T(n) = 5T(n/2) + n$
   - Algorithm 2: $T(n) = 3T(n/2) + n^2$

Which algorithm has the better asymptotic cost? Justify your answer by solving both recurrences (using recursion trees) and comparing the solutions.

2. A frog is jumping across a line of lily pads. It starts at lily pad 1. When the frog is at lily pad $i$ for any $i \geq 1$, it jumps to lily pad $i + 1$ with probability 1/2 and to lily pad $i + 2$ with probability 1/2.

   (a) Let $p(i)$ be the probability that the frog ever visits lily pad $i$, for any $i \geq 1$. Write a recurrence relation for $p(i)$. Don’t forget the base case(s).

   (b) Use annihilators to solve for a general solution to your recurrence relation.

   (c) Use the base case(s) of your recurrence to solve for an exact solution.

   (d) Now, let $X$ be a random variable giving the number of lily pads between lily pad 1 and $n$ that the frog visits, for some fixed number $n$. Compute $E(X)$ by using: linearity of expectation, indicator random variables, and your solution to the recurrence $p(i)$ that you found above.

3. Silly-Sort Consider the following sorting algorithm
Silly-Sort(A,i,j)
if A[i] > A[j]
then exchange A[i] and A[j];
if i+1 >= j
then return;
k = \lfloor(j-i+1)/3\rfloor;
Silly-Sort(A,i,j-k);
Silly-Sort(A,i+k,j);
Silly-Sort(A,i,j-k);

(a) Argue (by induction) that if n is the length of A, then Silly-
Sort(A,1,n) correctly sorts the input array A[1...n]
(b) Give a recurrence relation for the worst-case run time of Silly-Sort
and a tight bound on the worst-case run time
(c) Compare this worst-case runtime with that of insertion sort, merge
sort, heapsort and quicksort.

4. **Note:** In this problem, you’ll be writing - but not solving - a recurrence
relation over a data structure. When we get to dynamic programming
in class, we’ll see how to solve these types of recurrences.

Consider a rooted binary tree with nodes are labelled as follows. The
root node is labelled with the empty string. Then, any node that is a
left child of a node with name σ receives the name σL and any node
that is the right child of that node receives the name σR.

Give a recurrence relation returning the number of R’s in all labels of
all nodes. For example, the following tree has 10 R’s.

![Tree Diagram]

Hint: For a node v, let \( f(v) \) be the number of R’s in the tree rooted
at v, if the naming started at v. Also, let \( \ell(v) \) (resp. \( r(v) \)) be the left
(resp. right) child of $v$ if it exists or NULL otherwise. Finally, let $s(v)$ be the number of nodes in the subtree rooted at $v$ and assume this value is stored at each node. Now write a recurrence relation for $f(v)$. Don’t forget to include the base case and to test it on some examples.

A frog is jumping across a line of lily pads. It starts at lily pad 1. When the frog is at lily pad $i$ for any $i \geq 1$, it jumps to lily pad $i+1$ with probability $1/2$ and to lily pad $i + 2$ with probability $1/2$.

1. Let $p(i)$ be the probability that the frog ever visits lily pad $i$, for any $i \geq 1$. Write a recurrence relation for $p(i)$. Don’t forget the base case(s).

$$p(i) = 1/2p(i - 1) + 1/2p(i - 2)$$

2. Use annihilators to solve for a general solution to your recurrence relation.

Annihilator: $(L^2 - 1/2L - 1/2)$. General form: $p(n) = c_0(-1/2)^n + c_1$

3. Use the base case(s) of your recurrence to solve for an exact solution.

Specific form: $p(1) = 1$ and $p(2) = 1/2$. So we get $1 = -c_0/2 + c_1$ and $1/2 = c_0/4 + c_1$. Subtracting the second from the first, we get $1/2 = (-3/4)c_0$ or $c_0 = -2/3$. Then, plugging into the first equation, we get $1 = 1/3 + c_1$ or $c_1 = 2/3$. So the exact equation is: $p(n) = (-2/3)(-1/2)^n + 2/3$.

4. Now, let $X$ be a random variable giving the number of lily pads between lily pad 1 and $n$ that the frog visits, for some fixed number $n$. Compute $E(X)$ by using: linearity of expectation, indicator random variables, and your solution to the recurrence $p(i)$ that you found above.