CS 561, Randomized Data Structures: Skip Lists, Bloom Filters, Count-Min sketch

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Outline

- Skip Lists
- Bloom Filters
- Count-Min Sketch
A dictionary ADT implements the following operations

- *Insert(x)*: puts the item \( x \) into the dictionary
- *Delete(x)*: deletes the item \( x \) from the dictionary
- *IsIn(x)*: returns true iff the item \( x \) is in the dictionary
Skip List

- Enables insertions and searches for ordered keys in $O(\log n)$ expected time
- Very elegant randomized data structure, simple to code but analysis is subtle
- They guarantee that, with high probability, all the major operations take $O(\log n)$ time (e.g. Find-Max, Predecessor/Successor)
- Can even enable "find-i-th value" if store with each edge the number of elements that edge skips
Skip List

- A skip list is basically a collection of doubly-linked lists, $L_1, L_2, \ldots, L_x$, for some integer $x$
- Each list has a special head and tail node, the keys of these nodes are assumed to be $-\text{MAXNUM}$ and $+\text{MAXNUM}$ respectively
- The keys in each list are in sorted order (non-decreasing)
Skip List

- Every node is stored in the bottom list
- For each node in the bottom list, we flip a coin over and over until we get tails. For each heads, we make a duplicate of the node.
- The duplicates are stacked up in levels and the nodes on each level are strung together in sorted linked lists
- Each node $v$ stores a search key ($\text{key}(v)$), a pointer to its next lower copy ($\text{down}(v)$), and a pointer to the next node in its level ($\text{right}(v)$).
Example
Search

• To do a search for a key, $x$, we start at the leftmost node $L$
in the highest level
• We then scan through each level as far as we can without
  passing the target value $x$ and then proceed down to the next
  level
• The search ends either when we find the key $x$ or fail to find
  $x$ on the lowest level
Search

SkipListFind(x, L){
    v = L;
    while (v != NULL) and (Key(v) != x){
        if (Key(Right(v)) > x)
            v = Down(v);
        else
            v = Right(v);
    }
    return v;
}
Search Example
coin() returns "heads" with probability 1/2

Insert(k){
First call Search(k), let pLeft be the leftmost elem <= k in L_1
Insert k in List 0, to the right of pLeft
i = 1;
while (coin() = "heads"){
   insert k in List i;
   i++;
}
}
Deletion

- Deletion is very simple
- First do a search for the key to be deleted
- Then delete that key from all the lists it appears in from the bottom up, making sure to “zip up” the lists after the deletion
Analysis

- Intuitively, each level of the skip list has about half the number of nodes of the previous level, so we expect the total number of levels to be about $O(\log n)$
- Similarly, each time we add another level, we cut the search time in half except for a constant overhead
- So after $O(\log n)$ levels, we would expect a search time of $O(\log n)$
- We will now formalize these two intuitive observations
• For some key, \( k \), let \( X_k \) be the maximum height of \( k \) in the skip list.
• Q: What is the probability that \( X_k \geq 2 \log n \)?
• A: If \( p = \frac{1}{2} \), we have:

\[
P(X_k \geq 2 \log n) = \left(\frac{1}{2}\right)^{2\log n} = \frac{1}{(2^{\log n})^2} = \frac{1}{n^2}
\]

• Thus the probability that a particular key \( k \) achieves height \( 2 \log n \) is \( \frac{1}{n^2} \)
New Tool: Union Bound

FACT: Given two events $E_1$ and $E_2$,

$$Pr(E_1 \cup E_2) \leq Pr(E_1) + Pr(E_2)$$

Proof:

$$Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2) - Pr(E_1 \cap E_2)$$

$$\leq Pr(E_1) + Pr(E_2)$$

Generalizing to $n$ events, we have that:

$$Pr(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} Pr(E_i)$$
Height of Skip List

• Q: What is the probability that any key achieves height \(2 \log n\)?
• A: We want

\[
P(X_1 \geq 2 \log n \text{ or } X_2 \geq 2 \log n \text{ or } \ldots \text{ or } X_n \geq 2 \log n)
\]

• By a Union Bound, this probability is no more than

\[
P(X_1 \geq 2 \log n) + P(X_2 \geq 2 \log n) + \cdots + P(X_n \geq 2 \log n)
\]

• Which equals:

\[
\sum_{i=1}^{n} \frac{1}{n^2} = \frac{n}{n^2} = 1/n
\]
Height of Skip List

- This probability gets small as \( n \) gets large
- In particular, the probability of having a skip list of height exceeding \( 2 \log n \) is \( o(1) \)
- If an event occurs with probability \( 1 - o(1) \), we say that it occurs with high probability
- **Key Point:** The height of a skip list is \( O(\log n) \) with high probability.
A trick for computing expectations of discrete positive random variables:

- Let $X$ be a discrete r.v., that takes on values from 1 to $n$

$$E(X) = \sum_{i=1}^{n} P(X \geq i)$$
Why?

\[
\sum_{i=1}^{n} P(X \geq i) = P(X = 1) + P(X = 2) + P(X = 3) + \ldots + P(X = 2) + P(X = 3) + P(X = 4) + \ldots + P(X = 3) + P(X = 4) + P(X = 5) + \ldots + \ldots = 1Pr(X = 1) + 2Pr(X = 2) + 3Pr(X = 3) + \ldots = E(X)
\]
In-Class Exercise

Q: How much memory do we expect a skip list to use up?

- Let $X_k$ be the number of lists that key $k$ is inserted in.
- Q: What is $P(X_k \geq 1)$, $P(X_k \geq 2)$, $P(X_k \geq 3)$?
- Q: What is $P(X_k \geq i)$ for $i \geq 1$?
- Q: What is $E(X_k)$?
- Q: Let $X = \sum_{k=1}^{n} X_k$. What is $E(X)$?
Search Time

- It’s easier to analyze the search time if we imagine running the search backwards.
- Imagine that we start at the found node $v$ in the bottommost list and we trace the path backwards to the top leftmost sentinel, $L$.
- This will give us the length of the search path from $L$ to $v$ which is the time required to do the search.
Backwards Search

SLFback(v){
    while (v != L){
        if (Up(v)!=NIL)
            v = Up(v);
        else
            v = Left(v);
    }
}
Backward Search

- For every node \( v \) in the skip list \( \text{Up}(v) \) exists with probability 1/2. So for purposes of analysis, SLFBack is the same as the following algorithm:

```c
FlipWalk(v){
    while (v != L){
        if (COINFLIP == HEADS)
            v = Up(v);
        else
            v = Left(v);
    }
}
```
Analysis

- For this algorithm, the expected number of heads is exactly the same as the expected number of tails.
- Thus the expected run time of the algorithm is twice the expected number of upward jumps.
- Since we already know that the number of upward jumps is $O(\log n)$ with high probability, we can conclude that the expected search time is $O(\log n)$. 

Bloom Filters

- Randomized data structure for representing a set. Implements:
  - Insert(x) :
  - IsMember(x) :
- Allow false positives but require very little space
- Used frequently in: Databases, networking problems, p2p networks, packet routing
Bloom Filters

- Have $m$ slots, $k$ hash functions, $n$ elements; assume hash functions are all independent
- Each slot stores 1 bit, initially all bits are 0
- Insert($x$) : Set the bit in slots $h_1(x), h_2(x), ..., h_k(x)$ to 1
- IsMember($x$) : Return yes iff the bits in $h_1(x), h_2(x), ..., h_k(x)$ are all 1
Analysis Sketch

- $m$ slots, $k$ hash functions, $n$ elements; assume hash functions are all independent
- Then $P(\text{fixed slot is still 0}) = (1 - 1/m)^{kn}$
- Useful fact from Taylor expansion of $e^{-x}$:

$$e^{-x} - x^2/2 \leq 1 - x \leq e^{-x} \text{ for } x < 1$$

- Then if $x \leq 1$

$$e^{-x}(1 - x^2) \leq 1 - x \leq e^{-x}$$
Analysis

- Thus we have the following to good approximation.

\[
Pr(\text{fixed slot is still 0}) = (1 - 1/m)^{kn} 
\approx e^{-kn/m}
\]

- Let \( p = e^{-kn/m} \) and let \( \rho \) be the fraction of 0 bits after \( n \) elements inserted then

\[
Pr(\text{false positive}) = (1 - \rho)^k \approx (1 - p)^k
\]

- Where this last approximation holds because \( \rho \) is very close to \( p \) (by a Martingale argument beyond the scope of this class)
Analysis

- Want to minimize \((1-p)^k\), which is equivalent to minimizing 
  \[ g(p) = k \ln(1 - p) \]
- Trick: Note that 
  \[ g(p) = -(m/n) \ln(p) \ln(1 - p) \]
- By symmetry, this is minimized when \( p = 1/2 \) or equivalently 
  \[ k = (m/n) \ln 2 \]
- False positive rate is then \((1/2)^k \approx (.6185)^{m/n}\)
Tricks

- Can get the union of two sets by just taking the bitwise-or of the bit-vectors for the corresponding Bloom filters.
- Can easily half the size of a bloom filter - assume size is power of 2 then just bitwise-or the first and second halves together.
- Can approximate the size of the intersection of two sets - inner product of the bit vectors associated with the Bloom filters is a good approximation to this.
Extensions

- Bloomier Filters: Also allow for data to be inserted in the filter - similar functionality to hash tables but less space, and the possibility of false positives.
- A router forwards packets through a network
- A natural question for an administrator to ask is: what is the list of substrings of a fixed length that have passed through the router more than a predetermined threshold number of times
- This would be a natural way to try to, for example, identify worms and spam
- Problem: the number of packets passing through the router is *much* too high to be able to store counts for every substring that is seen!
Data Streams

- This problem motivates the data stream model
- Informally: there is a stream of data given as input to the algorithm
- The algorithm can take at most one pass over this data and must process it sequentially
- The memory available to the algorithm is much less than the size of the stream
- In general, we won’t be able to solve problems exactly in this model, only approximate
Our Problem

• We are presented with a stream of items $i$
• Want good approximation to the value $\text{Count}(i, T)$, which is the number of times we have seen item $i$ up to time $T$
Count-Min Sketch

• Our solution will be to use a data structure called a Count-Min Sketch
• This is a randomized data structure that will keep approximate values of $\text{Count}(i, T)$
• It is implemented using $k$ hash functions and $m$ counters
Count-Min Sketch

- Think of our \( m \) counters as being in a 2-dimensional array, with \( m/k \) counters per row and \( k \) rows
- Let \( C_{a,b} \) be the counter in row \( a \) and column \( b \)
- Our hash functions map items from the universe into counters
- In particular, hash function \( h_a \) maps item \( i \) to counter \( C_{a,h_a(i)} \)
Updates

- Initially all counters are set to 0
- When we see item $i$ in the data stream we do the following
- For each $1 \leq a \leq k$, increment $C_{a,h_a(i)}$
Count Approximations

- Let $C_{a,b}(T)$ be the value of the counter $C_{a,b}$ after processing $T$ tuples.
- We approximate $\text{Count}(i,T)$ by returning the value of the smallest counter associated with $i$.
- Let $m(i,T)$ be this value.
Analysis

**Theorem 1.** For any $\epsilon > 0$, with probability at least $1 - e^{-m\epsilon/e}$, our Count-Min Sketch ensures for every time step $T$ and every item $i$:

$$\text{Count}(i, T) \leq m(i, T) \leq \text{Count}(i, T) + \epsilon T$$
Proof

- Easy to see that $m(i, T) \geq \text{Count}(i, T)$, since each counter $C_{a,h_a(i)}$ incremented by $c_t$ every time pair $(i, c_t)$ is seen.
- Hard Part: Showing $m(i, T) \leq \text{Count}(i, T) + \epsilon T$.
- To see this, we will first consider the specific counter $C_{1,h_1(i)}$ and then use symmetry.
Proof

- Let $Z_1$ be a random variable giving the amount the counter is incremented by items other than $i$.
- Let $X_t$ be an indicator r.v. that is 1 if $j$ is the $t$-th item, and $j \neq i$ and $h_1(i) = h_1(j)$.
- Then $Z_1 = \sum_{t=1}^{T} X_t$.
- But if the hash functions are “good”, then if $i \neq j$, $Pr(h_1(i) = h_1(j)) = k/m$ (specifically, we need the hash functions to come from a 2-universal family, but we won’t get into that in this class).
- Hence, $E(X_t) = k/m$. 
Proof

• Thus, by linearity of expectation, we have that:

\[
E(Z_1) \leq \sum_{t=1}^{T} (k/m) = T k/m
\]

• We now need to make use of a very important inequality: Markov’s inequality
Markov’s Inequality

Markov’s Inequality. Let $X$ be a random variable that only takes on non-negative values. Then for any $\lambda > 0$:

$$\Pr(X \geq \lambda) \leq \frac{E(X)}{\lambda}$$

- Proof: Assume by contradiction that there exists a $\lambda$ such that $\Pr(X \geq \lambda)$ was actually larger than $E(X)/\lambda$
- But then the expected value of $X$ would be at least $\lambda \Pr(X \geq \lambda) > E(X)$, which is a contradiction!!!
Proof (Using Markov’s)

• Now, by Markov’s inequality,

\[ Pr(Z_1 \geq \epsilon T) \leq \frac{Tk/m}{\epsilon T} = \frac{k}{m\epsilon} \]

• This is the event where \( Z_1 \) is “bad” for item \( i \), in the sense that it gives more than a \( \epsilon T \) overestimate of how often item \( i \) has been seen.
Proof (Using Independence)

- Since our $k$ hash functions are “good” in the sense that they are independent, we have

$$\prod_{i=1}^{k} Pr(Z_j \geq \epsilon T) \leq \left(\frac{k}{m\epsilon}\right)^k$$
• Finally, we want to choose a $k$ that minimizes $f(k) = \left(\frac{k}{m\epsilon}\right)^k$

• Note that $\frac{\partial f}{\partial k} = \left(\frac{k}{m\epsilon}\right)^k (\ln \frac{k}{m\epsilon} + 1)$

• From this, we can see that the probability is minimized when $k = m\epsilon/e$, in which case:

$$\left(\frac{k}{m\epsilon}\right)^k = e^{-m\epsilon/e}$$
Recap

- Our Count-Min Sketch is very good at giving estimating counts of items with very little external space.
- Tradeoff is that it only provides approximate counts, but we can bound the approximation!
- Note: Can use the Count-Min Sketch to keep track of all the items in the stream that occur more than a given threshold ("heavy hitters")
- Basic idea is to store an item in a list of "heavy hitters" if its count estimate ever exceeds some given threshold.