CS 362, Randomized Data Structures: Skip Lists, Bloom Filters, Count-Min sketch

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Outline

• Skip Lists
• Bloom Filters
• Count-Min Sketch
Dictionary ADT

A dictionary ADT implements the following operations

- \textit{Insert}(x): puts the item x into the dictionary
- \textit{Delete}(x): deletes the item x from the dictionary
- \textit{IsIn}(x): returns true iff the item x is in the dictionary
Skip List

• Enables insertions and searches for ordered keys in $O(\log n)$ expected time
• Very elegant randomized data structure, simple to code but analysis is subtle
• They guarantee that, with high probability, all the major operations take $O(\log n)$ time (e.g. Find-Max, Predecessor/Successor)
• Can even enable ”find-i-th value” if store with each edge the number of elements that edge skips
Skip List

• A skip list is basically a collection of doubly-linked lists, $L_1, L_2, \ldots, L_x$, for some integer $x$
• Each list has a special head and tail node, the keys of these nodes are assumed to be $-\text{MAXNUM}$ and $+\text{MAXNUM}$ respectively
• The keys in each list are in sorted order (non-decreasing)
Skip List

- Every node is stored in the bottom list
- For each node in the bottom list, we flip a coin over and over until we get tails. For each heads, we make a duplicate of the node.
- The duplicates are stacked up in levels and the nodes on each level are strung together in sorted linked lists
- Each node \( v \) stores a search key (\( \text{key}(v) \)), a pointer to its next lower copy (\( \text{down}(v) \)), and a pointer to the next node in its level (\( \text{right}(v) \)).
Example
Search

- To do a search for a key, $x$, we start at the leftmost node $L$ in the highest level.
- We then scan through each level as far as we can without passing the target value $x$ and then proceed down to the next level.
- The search ends either when we find the key $x$ or fail to find $x$ on the lowest level.
Search

SkipListFind(x, L){
    v = L;
    while (v != NULL) and (Key(v) != x){
        if (Key(Right(v)) > x)
            v = Down(v);
        else
            v = Right(v);
    }
    return v;
}
Search Example
coin() returns "heads" with probability 1/2

Insert(k) {
First call Search(k), let pLeft be the leftmost elem <= k in L_1
Insert k in List 0, to the right of pLeft
i = 1;
while (coin() = "heads"){
    insert k in List i;
    i++;
}
}
Deletion

- Deletion is very simple
- First do a search for the key to be deleted
- Then delete that key from all the lists it appears in from the bottom up, making sure to “zip up” the lists after the deletion
• Intuitively, each level of the skip list has about half the number of nodes of the previous level, so we expect the total number of levels to be about $O(\log n)$
• Similarly, each time we add another level, we cut the search time in half except for a constant overhead
• So after $O(\log n)$ levels, we would expect a search time of $O(\log n)$
• We will now formalize these two intuitive observations
Height of Skip List

- For some key, $k$, let $X_k$ be the maximum height of $k$ in the skip list.
- Q: What is the probability that $X_k \geq 2 \log n$?
- A: If $p = 1/2$, we have:

$$P(X_k \geq 2 \log n) = \left(\frac{1}{2}\right)^{2 \log n} = \frac{1}{(2^{\log n})^2} = \frac{1}{n^2}$$

- Thus the probability that a particular key $k$ achieves height $2 \log n$ is $\frac{1}{n^2}$
New Tool: Union Bound

FACT: Given two events $E_1$ and $E_2$,

$$Pr(E_1 \cup E_2) \leq Pr(E_1) + Pr(E_2)$$

Proof:

$$Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2) - Pr(E_1 \cap E_2)$$

$$\leq Pr(E_1) + Pr(E_2)$$

Generalizing to $n$ events, we have that:

$$Pr(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} Pr(E_i)$$
Height of Skip List

• Q: What is the probability that any key achieves height 2 log n?
• A: We want

\[ P(X_1 \geq 2 \log n \text{ or } X_2 \geq 2 \log n \text{ or } \ldots \text{ or } X_n \geq 2 \log n) \]

• By a Union Bound, this probability is no more than

\[ P(X_1 \geq 2 \log n) + P(X_2 \geq 2 \log n) + \cdots + P(X_n \geq 2 \log n) \]

• Which equals:

\[ \sum_{i=1}^{n} \frac{1}{n^2} = \frac{n}{n^2} = 1/n \]
Height of Skip List

- This probability gets small as $n$ gets large
- In particular, the probability of having a skip list of height exceeding $2\log n$ is $o(1)$
- If an event occurs with probability $1 - o(1)$, we say that it occurs with high probability
- Key Point: The height of a skip list is $O(\log n)$ with high probability.
In-Class Exercise Trick

A trick for computing expectations of discrete positive random variables:

- Let $X$ be a discrete r.v., that takes on values from 1 to $n$

$$E(X) = \sum_{i=1}^{n} P(X \geq i)$$
Why?

\[ \sum_{i=1}^{n} P(X \geq i) = P(X = 1) + P(X = 2) + P(X = 3) + \ldots + P(X = 2) + P(X = 3) + P(X = 4) + \ldots + P(X = 3) + P(X = 4) + P(X = 5) + \ldots + \ldots = 1Pr(X = 1) + 2Pr(X = 2) + 3Pr(X = 3) + \ldots = E(X) \]
In-Class Exercise

Q: How much memory do we expect a skip list to use up?

• Let $X_k$ be the number of lists that key $k$ is inserted in.
• Q: What is $P(X_k \geq 1)$, $P(X_k \geq 2)$, $P(X_k \geq 3)$?
• Q: What is $P(X_k \geq i)$ for $i \geq 1$?
• Q: What is $E(X_k)$?
• Q: Let $X = \sum_{k=1}^{n} X_k$. What is $E(X)$?
Search Time

- It's easier to analyze the search time if we imagine running the search backwards.
- Imagine that we start at the found node $v$ in the bottommost list and we trace the path backwards to the top leftmost sentinel, $L$.
- This will give us the length of the search path from $L$ to $v$, which is the time required to do the search.
Backwards Search

```
SLFback(v){
    while (v != L){
        if (Up(v)!=NIL)
            v = Up(v);
        else
            v = Left(v);
    }
}
```
Backward Search

- For every node \( v \) in the skip list \( \text{Up}(v) \) exists with probability 1/2. So for purposes of analysis, \( \text{SLFBack} \) is the same as the following algorithm:

\[
\text{FlipWalk}(v)\{
\quad \text{while} \ (v \neq L)\{
\quad \quad \text{if} \ (\text{COINFLIP} == \text{HEADS})
\quad \quad \quad v = \text{Up}(v);
\quad \quad \text{else}
\quad \quad \quad v = \text{Left}(v);
\quad \}
\}
\]
Analysis

- For this algorithm, the expected number of heads is exactly the same as the expected number of tails.
- Thus the expected run time of the algorithm is twice the expected number of upward jumps.
- Since we already know that the number of upward jumps is $O(\log n)$ with high probability, we can conclude that the expected search time is $O(\log n)$. 
Bloom Filters

- Randomized data structure for representing a set. Implements:
  - Insert(x) :
  - IsMember(x) :
  - Allow false positives but require very little space
  - Used frequently in: Databases, networking problems, p2p networks, packet routing
Bloom Filters

- Have $m$ slots, $k$ hash functions, $n$ elements; assume hash functions are all independent
- Each slot stores 1 bit, initially all bits are 0
- Insert($x$) : Set the bit in slots $h_1(x), h_2(x), ..., h_k(x)$ to 1
- IsMember($x$) : Return yes iff the bits in $h_1(x), h_2(x), ..., h_k(x)$ are all 1
Analysis Sketch

- $m$ slots, $k$ hash functions, $n$ elements; assume hash functions are all independent
- Then $P(\text{fixed slot is still 0}) = (1 - 1/m)^{kn}$
- Useful fact from Taylor expansion of $e^{-x}$:
  \[ e^{-x} - x^2/2 \leq 1 - x \leq e^{-x} \text{ for } x < 1 \]
- Then if $x \leq 1$
  \[ e^{-x}(1 - x^2) \leq 1 - x \leq e^{-x} \]
Analysis

• Thus we have the following to good approximation.

\[ Pr(\text{fixed slot is still 0}) = (1 - 1/m)^{kn} \]
\[ \approx e^{-kn/m} \]

• Let \( p = e^{-kn/m} \) and let \( \rho \) be the fraction of 0 bits after \( n \) elements inserted then

\[ Pr(\text{false positive}) = (1 - \rho)^k \approx (1 - p)^k \]

• Where this last approximation holds because \( \rho \) is very close to \( p \) (by a Martingale argument beyond the scope of this class)
Analysis

- Want to minimize \((1 - p)^k\), which is equivalent to minimizing \(g(p) = k \ln(1 - p)\)
- Trick: Note that \(g(p) = -(m/n) \ln(p) \ln(1 - p)\)
- By symmetry, this is minimized when \(p = 1/2\) or equivalently \(k = (m/n) \ln 2\)
- False positive rate is then \((1/2)^k \approx (.6185)^{m/n}\)
Tricks

- Can get the union of two sets by just taking the bitwise-or of the bit-vectors for the corresponding Bloom filters
- Can easily half the size of a bloom filter - assume size is power of 2 then just bitwise-or the first and second halves together
- Can approximate the size of the intersection of two sets - inner product of the bit vectors associated with the Bloom filters is a good approximation to this.
Extensions

- Bloomier Filters: Also allow for data to be inserted in the filter - similar functionality to hash tables but less space, and the possibility of false positives.
Data Streams

- A router forwards packets through a network
- A natural question for an administrator to ask is: what is the list of substrings of a fixed length that have passed through the router more than a predetermined threshold number of times
- This would be a natural way to try to, for example, identify worms and spam
- Problem: the number of packets passing through the router is *much* too high to be able to store counts for every substring that is seen!
Data Streams

- This problem motivates the data stream model
- Informally: there is a stream of data given as input to the algorithm
- The algorithm can take at most one pass over this data and must process it sequentially
- The memory available to the algorithm is much less than the size of the stream
- In general, we won’t be able to solve problems exactly in this model, only approximate
Our Problem

• We are presented with a stream of items $i$
• Want good approximation to the value $\text{Count}(i, T)$, which is the number of times we have seen item $i$ up to time $T$
Count-Min Sketch

- Our solution will be to use a data structure called a *Count-Min Sketch*
- This is a randomized data structure that will keep approximate values of $\text{Count}(i, T)$
- It is implemented using $k$ hash functions and $m$ counters
Count-Min Sketch

- Think of our \( m \) counters as being in a 2-dimensional array, with \( m/k \) counters per row and \( k \) rows
- Let \( C_{a,b} \) be the counter in row \( a \) and column \( b \)
- Our hash functions map items from the universe into counters
- In particular, hash function \( h_a \) maps item \( i \) to counter \( C_{a,h_a(i)} \)
Updates

- Initially all counters are set to 0
- When we see item $i$ in the data stream we do the following
- For each $1 \leq a \leq k$, increment $C_{a,h_a(i)}$
Count Approximations

- Let $C_{a,b}(T)$ be the value of the counter $C_{a,b}$ after processing $T$ tuples
- We approximate Count($i,T$) by returning the value of the smallest counter associated with $i$
- Let $m(i,T)$ be this value
Analysis

**Theorem 1.** For any $\epsilon > 0$, with probability at least $1 - e^{-m\epsilon/e}$, our Count-Min Sketch ensures for every time step $T$ and every item $i$:

$$\text{Count}(i, T) \leq m(i, T) \leq \text{Count}(i, T) + \epsilon T$$
Proof

• Easy to see that $m(i, T) \geq \text{Count}(i, T)$, since each counter $C_{a,h_a(i)}$ incremented by $c_t$ every time pair $(i, c_t)$ is seen
• Hard Part: Showing $m(i, T) \leq \text{Count}(i, T) + \epsilon T$.
• To see this, we will first consider the specific counter $C_{1,h_1(i)}$ and then use symmetry.
Proof

• Let $Z_1$ be a random variable giving the amount the counter is incremented by items other than $i$
• Let $X_t$ be an indicator r.v. that is 1 if $j$ is the $t$-th item, and $j \neq i$ and $h_1(i) = h_1(j)$
• Then $Z_1 = \sum_{t=1}^{T} X_t$
• But if the hash functions are “good”, then if $i \neq j$, $Pr(h_1(i) = h_1(j)) = k/m$ (specifically, we need the hash functions to come from a 2-universal family, but we won’t get into that in this class)
• Hence, $E(X_t) = k/m$
Proof

- Thus, by linearity of expectation, we have that:

\[ E(Z_1) \leq \sum_{t=1}^{T} \frac{k}{m} \]

\[ = T \frac{k}{m} \]

- We now need to make use of a very important inequality: Markov's inequality
Markov’s Inequality

**Markov’s Inequality.** Let $X$ be a random variable that only takes on non-negative values. Then for any $\lambda > 0$:

$$Pr(X \geq \lambda) \leq E(X)/\lambda$$

- Proof: Assume by contradiction that there exists a $\lambda$ such that $Pr(X \geq \lambda)$ was actually larger than $E(X)/\lambda$
- But then the expected value of $X$ would be at least $\lambda Pr(X \geq \lambda) > E(X)$, which is a contradiction!!!
Proof (Using Markov’s’s)

• Now, by Markov’s inequality,

\[ Pr(Z_1 \geq \epsilon T) \leq \frac{Tk/m}{\epsilon T} = \frac{k}{m\epsilon} \]

• This is the event where \( Z_1 \) is “bad” for item \( i \), in the sense that it gives more than a \( \epsilon T \) overestimate of how often item \( i \) has been seen.
Proof (Using Independence)

- Since our $k$ hash functions are “good” in the sense that they are independent, we have

\[
\prod_{i=1}^{k} \Pr(Z_j \geq \epsilon T) \leq \left(\frac{k}{m\epsilon}\right)^k
\]
Proof (Choosing $k$)

- Finally, we want to choose a $k$ that minimizes $f(k) = \left( \frac{k}{m\epsilon} \right)^k$
- Note that $\frac{\partial f}{\partial k} = \left( \frac{k}{m\epsilon} \right)^k \left( \ln \frac{k}{m\epsilon} + 1 \right)$
- From this, we can see that the probability is minimized when $k = m\epsilon/e$, in which case:

$$\left( \frac{k}{m\epsilon} \right)^k = e^{-m\epsilon/e}$$
Recap

- Our Count-Min Sketch is very good at giving estimating counts of items with very little external space.
- Tradeoff is that it only provides approximate counts, but we can bound the approximation!
- Note: Can use the Count-Min Sketch to keep track of all the items in the stream that occur more than a given threshold ("heavy hitters").
- Basic idea is to store an item in a list of "heavy hitters" if its count estimate ever exceeds some given threshold.