Today's Outline

“The path that can be trodden is not the enduring and unchanging Path. The name that can be named is not the enduring and unchanging Name.” - Tao Te Ching

- Bellman-Ford Wrapup
- All-Pairs Shortest Paths

InitSSSP

```
InitSSSP(s)
{
    dist(s) = 0;
    pred(s) = NULL;
    for all vertices v != s{
        dist(v) = infinity;
        pred(v) = NULL;
    }
}
```

GenericSSSP

```
GenericSSSP(s){
    InitSSSP(s);
    put s in the bag;
    while the bag is not empty{
        take u from the bag;
        for all edges (u,v){
            if (u,v) is tense{
                Relax(u,v);
                put v in the bag;
            }
        }
    }
}
```
Bellman-Ford

- If we replace the bag in the GenericSSSP with a queue, we get the Bellman-Ford algorithm
- Bellman-Ford is efficient even if there are negative edges and it can be used to quickly detect the presence of negative cycles
- If there are no negative edges, however, Dijkstra’s algorithm is faster than Bellman-Ford

Invariant

- A simple inductive argument (left as an exercise) shows the following invariant:
- At the end of the i-th phase, for each vertex v, dist(v) is less than or equal to the length of the shortest path s \rightarrow v consisting of i or fewer edges
- This implies that the algorithm ends in O(|V|) phases

Analysis

- The easiest way to analyze this algorithm is to break the execution into phases
- Before we begin the alg, we insert a token into the queue
- Whenever we take the token out of the queue, we begin a new phase by just reinserting the token into the queue
- The 0-th phase consists entirely of scanning the source vertex s
- The algorithm ends when the queue contains only the token

Example

Four phases of the Bellman-Ford algorithm run on a directed graph with negative edges. Nodes are taken from the queue in the order s \diamond a b c \diamond d f b \diamond a e d \diamond d a \diamond \diamond, where \diamond is the token. Shaded vertices are in the queue at the end of each phase. The bold edges describe the evolving shortest path tree.
Since a shortest path can only pass through each vertex once, either the algorithm halts before the $|V|$-th phase or the graph contains a negative cycle.

In each phase, we scan each vertex at most once and so we relax each edge at most once.

Hence the run time of a single phase is $O(|E|)$.

Thus, the overall run time of Bellman-Ford is $O(|V||E|)$.

Now that we understand how the phases of Bellman-Ford work, we can simplify the algorithm. Instead of using a queue to perform a partial BFS in each phase, we will just scan through the adjacency list directly and try to relax every edge in the graph.

This will be much closer to how the textbook presents Bellman-Ford.

The run time will still be $O(|V||E|)$.

To show correctness, we’ll have to show that an invariant holds which can be proved by induction on $i$.

Dijkstra’s algorithm and Bellman-Ford are both variants of the GenericSSSP algorithm for solving SSSP.

Dijkstra’s algorithm uses a Fibonacci heap for the bag while Bellman-Ford uses a queue.

Dijkstra’s algorithm runs in time $O(|E| + |V| \log |V|)$ if there are no negative edges.

Bellman-Ford runs in time $O(|V||E|)$ and can handle negative edges (and detect negative cycles).
All-Pairs Shortest Paths

- For the single-source shortest paths problem, we wanted to find the shortest path from a source vertex \( s \) to all the other vertices in the graph
- We will now generalize this problem further to that of finding the shortest path from every possible source to every possible destination
- In particular, for every pair of vertices \( u \) and \( v \), we need to compute the following information:
  - \( \text{dist}(u, v) \) is the length of the shortest path (if any) from \( u \) to \( v \)
  - \( \text{pred}(u, v) \) is the second-to-last vertex (if any) on the shortest path (if any) from \( u \) to \( v \)

Example

- For any vertex \( v \), we have \( \text{dist}(v, v) = 0 \) and \( \text{pred}(v, v) = \text{NULL} \)
- If the shortest path from \( u \) to \( v \) is only one edge long, then \( \text{dist}(u, v) = w(u \rightarrow v) \) and \( \text{pred}(u, v) = u \)
- If there’s no shortest path from \( u \) to \( v \), then \( \text{dist}(u, v) = \infty \) and \( \text{pred}(u, v) = \text{NULL} \)

Lots of Single Sources

- The output of our shortest path algorithm will be a pair of \(|V| \times |V|\) arrays encoding all \(|V|^2\) distances and predecessors.
- Many maps contain such a distance matrix - to find the distance from (say) Albuquerque to (say) Ruidoso, you look in the row labeled “Albuquerque” and the column labeled “Ruidoso”
- In this class, we'll focus only on computing the distance array
- The predecessor array, from which you would compute the actual shortest paths, can be computed with only minor additions to the algorithms presented here

APSP

- Most obvious solution to APSP is to just run SSSP algorithm \(|V|\) times, once for every possible source vertex
- Specifically, to fill in the subarray \( \text{dist}(s, \cdot) \), we invoke either Dijkstra’s or Bellman-Ford starting at the source vertex \( s \)
- We’ll call this algorithm ObviousAPSP
ObviousAPSP

```
ObviousAPSP(V,E,w){
    for every vertex s{
        dist(s,*) = SSSP(V,E,w,s);
    }
}
```

Analysis

- The running time of this algorithm depends on which SSSP algorithm we use
- If we use Bellman-Ford, the overall running time is $O(|V|^2|E|) = O(|V|^4)$
- If all the edge weights are positive, we can use Dijkstra’s instead, which decreases the run time to $\Theta(|V||E| + |V|^2 \log |V|) = O(|V|^3)$

Problem

- We’d like to have an algorithm which takes $O(|V|^3)$ but which can also handle negative edge weights
- We’ll see that a dynamic programming algorithm, the Floyd Warshall algorithm, will achieve this
- Note: the book discusses another algorithm, Johnson’s algorithm, which is asymptotically better than Floyd Warshall on sparse graphs. However we will not be discussing this algorithm in class.

Dynamic Programming

- Recall: Dynamic Programming = Recursion + Memorization
- Thus we first need to come up with a recursive formulation of the problem
- We might recursively define $dist(u,v)$ as follows:

$$
\begin{cases} 
0 & \text{if } u = v \\
\min_x (dist(u,x) + w(x \rightarrow v)) & \text{otherwise}
\end{cases}
$$
The problem

- In other words, to find the shortest path from $u$ to $v$, try all possible predecessors $x$, compute the shortest path from $u$ to $x$ and then add the last edge $u \rightarrow v$
- Unfortunately, this recurrence doesn't work
- To compute $\text{dist}(u, v)$, we first must compute $\text{dist}(u, x)$ for every other vertex $x$, but to compute any $\text{dist}(u, x)$, we first need to compute $\text{dist}(u, v)$
- We're stuck in an infinite loop!

The Recurrence

$\text{dist}(u, v, k) = \begin{cases} 
0 & \text{if } u = v \\
\infty & \text{if } k = 0 \text{ and } u \neq v \\
\min_x \left( \text{dist}(u, x, k - 1) + w(x \rightarrow v) \right) & \text{otherwise}
\end{cases}$

The solution

- To avoid this circular dependency, we need some additional parameter that decreases at each recursion and eventually reaches zero at the base case
- One possibility is to include the number of edges in the shortest path as this third magic parameter
- So define $\text{dist}(u, v, k)$ to be the length of the shortest path from $u$ to $v$ that uses at most $k$ edges
- Since we know that the shortest path between any two vertices uses at most $|V| - 1$ edges, what we want to compute is $\text{dist}(u, v, |V| - 1)$

The Algorithm

- It's not hard to turn this recurrence into a dynamic programming algorithm
- Even before we write down the algorithm, though, we can tell that its running time will be $\Theta(|V|^4)$
- This is just because the recurrence has four variables — $u$, $v$, $k$ and $x$ — each of which can take on $|V|$ different values
- Except for the base cases, the algorithm will just be four nested “for” loops
DP-APSP

\[
\text{DP-APSP}(V,E,w)\{
\text{for all vertices } u \text{ in } V\{
\text{for all vertices } v \text{ in } V\{
\text{if } (u=v) \text{ then } \text{dist}(u,v,0) = 0; \\
\text{else } \text{dist}(u,v,0) = \infty; \\
} \}
\}\}\}
\]

\[
\text{for } k=1 \text{ to } |V|-1\{
\text{for all vertices } u \text{ in } V\{
\text{for all vertices } v \text{ in } V\{
\text{dist}(u,v,k) = \infty; \\
\text{for all vertices } x \text{ in } V\{
\text{if } \text{dist}(u,v,k) > \text{dist}(u,x,k-1)+w(x,v) \\
\text{dist}(u,v,k) = \text{dist}(u,x,k-1)+w(x,v); \\
} \}} \}
\}\}
\]

Floyd-Warshall

\[
\text{Floyd-Warshall}\{
\text{Number the vertices arbitrarily from 1 to } |V|\}
\text{Define } \text{dist}(u,v,r) \text{ to be the shortest path from } u \text{ to } v \text{ where all intermediate vertices (if any) are numbered } r \text{ or less} \\
\text{If } r = 0, \text{ we can't use any intermediate vertices so shortest path from } u \text{ to } v \text{ is just the weight of the edge (if any) between } u \text{ and } v \\
\text{If } r > 0, \text{ then either the shortest legal path from } u \text{ to } v \text{ goes through vertex } r \text{ or it doesn't} \\
\text{We need to compute the shortest path distance from } u \text{ to } v \text{ with no restrictions, which is just } \text{dist}(u,v,|V|) \\
\]\n
\[
\text{The Problem}\{
\text{This algorithm still takes } O(|V|^4) \text{ which is no better than the ObviousAPSP algorithm} \\
\text{If we use a certain divide and conquer technique, there is a way to get this down to } O(|V|^3 \log |V|) \text{ (think about how you might do this)} \\
\text{However, to get down to } O(|V|^3) \text{ run time, we need to use a different third parameter in the recurrence} \\
\}\}

\[
\text{The recurrence}\{
\text{We get the following recurrence:} \\
\text{dist}(u,v,r) = \begin{cases} \\
\quad w(u \rightarrow v) & \text{if } r = 0 \\
\quad \min\{\text{dist}(u,v,r-1), \text{dist}(u,r,r-1)+\text{dist}(r,v,r-1)\} & \text{otherwise} \\
\end{cases}
\}\}
The Algorithm

\[
\text{FloydWarshall}(V,E,w)\{
\text{for } u=1 \text{ to } |V|\{
    \text{for } v=1 \text{ to } |V|\{
        \text{dist}(u,v,0) = w(u,v);
    }\}
\text{for } r=1 \text{ to } |V|\{
    \text{for } u=1 \text{ to } |V|\{
        \text{for } v=1 \text{ to } |V|\{
            \text{if (dist}(u,v,r-1) < \text{dist}(u,r,r-1) + \text{dist}(r,v,r-1))
                \text{dist}(u,v,r) = \text{dist}(u,v,r-1);
            \text{else}
                \text{dist}(u,v,r) = \text{dist}(u,r,r-1) + \text{dist}(r,v,r-1);
        }\}
    }\}
}\}
\]

Analysis

- There are three variables here, each of which takes on \(|V|\) possible values
- Thus the run time is \(\Theta(|V|^3)\)
- Space required is also \(\Theta(|V|^3)\)

Take Away

- Floyd-Warshall solves the APSP problem in \(\Theta(|V|^3)\) time even with negative edge weights
- Floyd-Warshall uses dynamic programming to compute APSP
- We’ve seen that sometimes for a dynamic program, we need to introduce an \textit{extra variable} to break dependencies in the recurrence.
- We’ve also seen that the choice of this extra variable can have a big impact on the run time of the dynamic program