CS 506, HW1

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Due: Feb. 20th

You are encouraged to work on the homework in groups of about 2 or 3. You may turn in one writeup per group, but please certify that all members worked on each problem. Note that David Several of these problems are from the book "Computational Geometry (third edition)" by Berg, et al. Mount's notes and homework problems are available in the link off the course web page: http://www.cs.umd.edu/class/fall2016/cmsc754/Handouts/cmsc754-2016-08-handouts.pdf

1. (Exercise 8.1 from Berg) Prove that the duality transform discussed in class is indeed incidence and order reversing.

2. (Exercise 8.2 from Berg) The dual of a line segment is a left-right double wedge, as discussed in class. Answer the following

   (a) What is the dual of the collection of points inside a given triangle with vertices p,q and r

   (b) What type of object in the primal plane would dualize to a top-bottom double wedge?

3. In the online convex hull problem, we are given a set of n points one at a time. After receiving each point, we compute the convex hull of all points seen so far. Consider this problem in the 2D plane. Give an efficient online algorithm to update the convex hull when a new point is given. Analyze your algorithm.

4. Problem 2, HW 1 from David Mount's class (Pareto Optimal/Convex Hull problem) quoted below.
   Consider a set of points $P = \{p_1, \dots p_n\}$ in the plane where $p_i = (x_i, y_i)$. A *Pareto set* for P, denoted $Pareto(P)$ is a subset of points P' such that for each $p_i \in P'$, there is no $p_j \in P$ such that $x_j \geq x_i$ and $y_j \geq y_i$. That is, each point of $Pareto(P)$ has the property that there

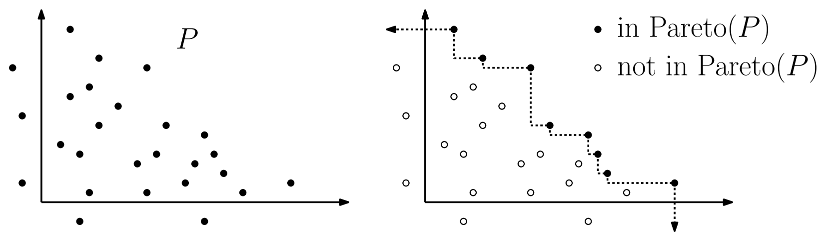

Figure 1. Example Pareto optimal figure (from David Mount hw).

is no point of P that is both to the right and above it. Pareto sets are important whenever you want to optimize *two* criteria (e.g. accuracy and precision of a machine learning algorithm, cheapness and flight "shortness" for airline tickets, etc.), since they represent the optimal "envelope" of possible solutions.

This problem explores the many similarities between Pareto sets and convex hulls. Whenever a problem asks for an algorithm, briefly justify correctness of your algorithm, explain any non-standard data structures, and derive the runtime.

(a) A point p lies on the convex hull of a set P if and only if there is a line passing through p such that all the points of P lie on one side of this line. Provide an analogous assertion for the points of $Pareto(P)$ in terms of a different shape.

(b) Devise an analogue of Graham's convex-hull algorithm for computing $Pareto(P)$ in $O(n \log n)$ time. Briefly justify your algorithm's correctness and derive its runtime. (You don't need to explain the algorithm "from scratch"; you can explain what modifications need to be made to Graham's algorithm.

(c) Devise an analogue of Jarvis march algorithm for computing $Paret(P)$ in $O(hn)$ time where h is the cardinality of $Pareto(P)$. (as in the last part, you can just explain the differences with Jarvis's algorithm.

(d) Devise an algorithm for computing $Pareto(P)$ in $O(n \log h)$ time. Hint: Chan!

5. (Exercise 8.7 from Berg) Let R be a set of n red points in the plane, and let B be a set of n points in the plane. A **separator** or R and B is a line ℓ that has all points of R to one side and all points of B to the other. Give a randomized algorithm that can decide in $O(n)$ expected

time whether R and B have a separator. Hint: A deterministic $O(n^2)$ algorithm is easy. To do better, use the dual and arrangements.

6. (Exercise 8.10 from Berg) Let L be a set of n non-vertical lines in the plane. Suppose that the arrangement $\mathcal{A}(L)$ only has vertices with level 0. What can you say about this arrangement? Next suppose that lines of L can be vertical. What can you say now about the arrangement?

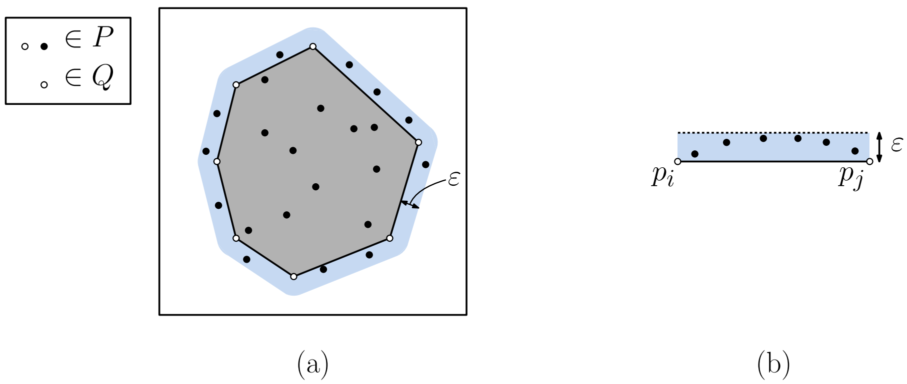

Figure 2. Computing an ϵ-sketch (from David Mount hw)

7. Problem 4, HW 3 from David Mount's class (ϵ sketch of convex hull), quoted below.
   You are given a set P of n points lying in the unit square in the plane. Given any subset $Q \subseteq P$, clearly we have $\text{conv}(Q) \subseteq \text{conv}(P)$. For $\epsilon > 0$, we say that Q is an ϵ-sketch of P if every point of P lies within distance at most ϵ of $\text{conv}(Q)$ (See Figure 2 (a)).

   (a) Consider the following simple greedy algorithm for computing an ϵ-sketch of a planar point set P. First let $\langle p_0, \ldots, p_{k-1} \rangle$ denote the vertices of $\text{conv}(P)$ listed in counterclockwise order. Put p_0 in Q and set $i \leftarrow 0$. Find the largest index j, $i < j \leq k$ such that all the points $\{p_{i+1}, \ldots p_{j-1}$ lie within distance ϵ of the line segment p_i, p_j (See Fig 2(b)). (Indices are taken modulo k so $p_k = p_0$.) If $j = k$ then stop. Otherwise, add p_j to Q, set $i \leftarrow j$ and repeat. Show that this procedure correctly produces an ϵ-sketch of P.

   (b) Fix some $\epsilon > 0$. Let $m(P)$ be the minimum number of points on any ϵ-sketch of P. Assume the points of P are in convex position, i.e. they all lie on the border of $\text{conv}(P)$. Show that the greedy procedure above produces a sketch of size at most $m(P) + 1$. Hint: First show that the points in any minimum sketch can be assumed to be taken from the vertices of the convex hull of P. Second, consider the points of the minimum sketch in cyclic order about P's convex hull and ask how many points generated by the greedy algorithm can be created between any two points of the minimum sketch.

(c) Prove that $m(P) = O(1/\epsilon)$. I.e., that it it is completely independent of the number of vertices of the convex hull. What implication does this have on the amount of space needed to store an ϵ-approximation of all crepe recipes?

8. Challenge: (This is the type of problem that could turn into a project or potentially a paper) Can you adapt the ϵ-sketch convex hull problem to come up with a similar type of sketch of the upper envelope in an arrangement? What can you say formally about the number of lines in your sketch of the upper envelope and how well the sketch approximates the true upper envelope? (Super Challenge: Any connections to sketching a Voronoi diagram?)