CS 506, HW 2

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Due: March 10th

You are encouraged to work on the homework in groups of about 2 or 3. You may turn in one writeup per group, but please certify that all members worked on each problem.

1. Let $P$ be a set of $n$ points in $\mathbb{R}^2$. Show that $P$ can be preprocessed in $O(n^2 \log n)$ time to create a data structure that can answer any query: “How many points lie below a query line?” in $O(\log n)$ time.

2. The following question is about maximum degree vertices in both Voronoi diagrams and triangulations.

   (a) Prove that for any $n > 3$ there is a set of $n$ points in the plane such that one of the cells of the Voronoi diagram of these points has $n - 1$ vertices.

   (b) The degree of a point in a triangulation is the number of edges incident to it. Give an example of a set of $n$ points in the plane, such that, no matter how the set is triangulated, there is always a point whose degree is $n - 1$.

3. Recall that a 5-clique is a graph of 5 nodes that are all completely connected

   (a) Prove that you can not draw a 5-clique in the plane with no edge crossings.

   (b) The Euler characteristic of a torus is 0 (recall this means that $V - E + F = 0$, for any graph drawn on a torus, where $V$ is vertices, $E$ is edges and $F$ is faces). Can you draw a 5-clique with no edge crossings on a torus?

   (c) The graph $K_{3,3}$ is the complete bipartite graph with 3 nodes on both sides. It’s the 3 home, 3 utility graph that we proved can’t
be drawn with no edge crossings in the plane. A Mobius strip\(^1\) has Euler characteristic 0. Can you draw \(K_{3,3}\) with no edge crossings on a Mobius strip? If so, please demonstrate (google Moibus strip for how to make one with tape and scissors). If not, prove it is impossible.

4. (Exercise 9.16 from Berg) A \(k\)-clustering of a set \(P\) of \(n\) points in the plane is a partitioning of \(P\) into \(k\) non-empty subsets \(P_1, \ldots, P_k\). Define the distance between any pair \(P_i, P_j\) of clusters to be the minimum distance between one point from \(P_i\) and one point from \(P_j\) that is:

\[
\text{dist}(P_i, P_j) = \min_{p \in P_i, q \in P_j} \text{dist}(p, q)
\]

We want to find a \(k\)-clustering (for given \(k\) and \(P\)) that maximizes the minimum distance between clusters.

(a) Suppose the minimum distance between clusters is achieved by point \(p \in P_i\) and \(q \in P_j\). Prove that \(pq\) is an edge of the Delaunay triangulation of \(P\).

(b) Give an \(O(n \log n)\) time algorithm to compute a \(k\)-clustering maximizing the minimum distance between clusters. Hint: Use a Union-Find data structure!

5. (Adapted from Mount F’16) There is a set of \(n\) building tops, represented by points \(P = \{p_1, \ldots, p_n\}\) and \(m\) floating balloons, represented by points \(Q = \{q_1, \ldots, q_m\}\) (Figure 1). You have a cannon in \(\mathbb{R}^2\) that has three controls labeled “a”, “b”, and “c”. A projectile shot from this cannon travels along the arc \(y = a + bx - cx^2\). Can you adjust

\[y = a + bx - cx^2\]

\[
Q
\]

\[
P
\]

\[\text{Figure 1. LP}\]

\(^1\)Fun fact: Mobius strips are used in conveyor belts to ensure the entire surface area of the belt gets even wear.
the cannon so that the projectile travels above the set $P$, but below the set $Q$? You should determine this in $O(n + m)$ time. You can assume anything about the initial location of the cannon so long as you clearly state it. Hint: Use Linear Programming. Just FYI: This has applications to learning a quadratic classifier that separates $P$ and $Q$.

6. You are given a set of points $P$ in the plane. Your goal is to find the smallest circle that contains all points. Give an efficient algorithm to do this. Hint: This can be done using an incremental algorithm and backwards analysis as we discussed in class for Siedel’s LP algorithm.

7. Write a brief proposal for your class project (1 page). Please talk to me about your ideas briefly before you start writing this.