CS 506, HW 2

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Due: March 10th

You are encouraged to work on the homework in groups of about 2 or 3. You may turn in one writeup per group, but please certify that all members worked on each problem.

1. Let P be a set of n points in $\mathbb{R}^2$. Show that P can be preprocessed in $O(n^2 \log n)$ time to create a data structure that can answer any query: "How many points lie below a query line?" in $O(\log n)$ time.

2. The following question is about maximum degree vertices in both Voronoi diagrams and triangulations.

   (a) Prove that for any $n > 3$ there is a set of n points in the plane such that one of the cells of the Voronoi diagram of these points has $n - 1$ vertices.

   (b) The degree of a point in a triangulation is the number of edges incident to it. Give an example of a set of n points in the plane, such that, no matter *how* the set is triangulated, there is always a point whose degree is $n - 1$.

3. Recall that a 5-clique is a graph of 5 nodes that are all completely connected

   (a) Prove that you can not draw a 5-clique in the plane with no edge crossings.

   (b) The Euler characteristic of a torus is 0 (recall this means that $V - E + F = 0$, for any graph drawn on a torus, where V is vertices, E is edges and F is faces). Can you draw a 5-clique with no edge crossings on a torus?

   (c) The graph $K_{3,3}$ is the complete bipartite graph with 3 nodes on both sides. It's the 3 home, 3 utility graph that we proved can't

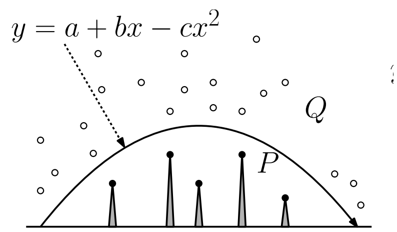

Figure 1. LP

be drawn with no edge crossings in the plane. A Mobius strip[1] has Euler characteristic 0. Can you draw $K_{3,3}$ with no edge crossings on a Moibus strip? If so, please demonstrate (google Moibus strip for how to make one with tape and scissors). If not, prove it is impossible.

4. (Exercise 9.16 from Berg) A k-clustering of a set P of n points in the the plane is a partitioning of P into k non-empty subsets $P_1, \ldots, P_k$. Define the distance between any pair P_i,P_j of clusters to be the minimum distance between one point from P_i and one point from P_j that is:

$$dist(P_i, P_j) = \min_{p \in P_i, q \in P_j} dist(p, q)$$

We want to find a k-clustering (for given k and P) that maximizes the minimum distance between clusters.

   (a) Suppose the minimum distance between clusters is achieved by point $p \in P_i$ and $q \in P_j$. Prove that $\overline{pq}$ is an edge of the Delaunay triangulation of P

   (b) Give an $O(n \log n)$ time algorithm to compute a k-clustering maximizing the minimum distance between clusters. Hint: Use a Union-Find data structure!

5. (Adapted from Mount F'16) There is a set of n building tops, represented by points $P = \{p_1, \ldots p_n\}$ and m floating balloons, represented by points $Q = \{q_1, \ldots q_m\}$ (Figure 1). You have a cannon in $\mathbb{R}^2$ that has three controls labeled "a", "b", and "c". A projectile shot from this cannon travels along the arc $y = a + bx - cx^2$. Can you adjust

[1] Fun fact: Mobius strips are used in conveyer belts to ensure the entire surface area of the belt gets even wear.

the cannon so that the projectile travels above the set P, but below the set Q? You should determine this in $O(n + m)$ time. You can assume anything about the initial location of the canon so long as you clearly state it. Hint: Use Linear Programming. Just FYI: This has applications to learning a quadratic classifier that separates P and Q.

6. You are given a set of points P in the plane. Your goal is to find the smallest circle that contains all points. Give an efficient algorithm to do this. Hint: This can be done using an incremental algorithm and backwards analysis as we discussed in class for Siedel's LP algorithm.

7. Write a brief proposal for your class project (1 page). Please talk to me about your ideas briefly before you start writing this.