1 Halfplane Intersection Problem

We can represent lines in a plane by the equation

\[ y = ax + b \]

where \( a \) is the slope and \( b \) the y-intercept. Note that this can not represent vertical lines, which have infinite slope (but these do not occur in general position).

Every non-vertical line defines two halfplanes, consisting of the points above and below the line.

- **lower halfplane**: \( y \leq ax + b \)
- **upper halfplane**: \( y \geq ax + b \)

1.1 Halfplane Intersection

**Halfplane Intersection Problem**: Given a collection \( H = \{h_1, \ldots, h_n\} \) of \( n \) closed halfplanes, compute their intersection.

Note that a halfplane is a convex set so the intersection of any number of them is also convex. Unlike the convex hull, the intersection of halfplanes may be empty or unbounded.

A good way to represent the output would be to list the lines bounding the intersection in clockwise order.

How big can this output be? By convexity, each halfplane can appear only once as a side. Hence number of sides is \( O(n) \).

Lower bound? Can show via sorting it is \( \Omega(n \log n) \). [Jared: good exercise]

Motivation: Generate convex shape approximations. Optimizations problems (e.g. linear programming, gradient descent) often occur over intersection of halfspaces.

Higher dimensions: This can be challenging since the number of facets in the intersection of \( n \) halfspaces in \( d \)-dimensions can be \( \Theta(n^{\lfloor d/2 \rfloor}) \).

We’ll discuss two algorithms: (1) divide-and-conquer; and (2) duality based.

2 Divide-and-Conquer

**procedure** `DIVIDE-AND-CONQUER(H)`

If \( n = |H| = 1 \), return the halfplane

Partition the halfplanes into \( H_1 \) and \( H_2 \) of size \( \lfloor n/2 \rfloor \) and \( \lceil n/2 \rceil \)

Let \( K_1 \) and \( K_2 \) be the intersections of \( H_1 \) and \( H_2 \), computed recursively

Return the intersection of \( K_1 \) and \( K_2 \)

end procedure

If we can perform the intersection (last step) in \( O(n) \) time then the algorithm will take \( O(n \log n) \) time, which can be shown via solving a simple recurrence \( T(n) = 2T(n/2) + n \).

Thus, we must intersect two convex (possibly empty or unbounded) polygons in linear time.
2.1 Plane Sweep

We compute the intersection of $K_1$ and $K_2$ via a plane sweep.

First, break both polygons into upper and lower chains. The upper chain of a polygon is just the sequence of points that are above the line induced by the leftmost and rightmost point in the polygon. The lower chain is the sequence of points below this line. So this step can be done in $O(n)$ time.

Next, sweep! By convexity, the sweep line intersects each polygon in at most two points. Hence, there are at most 4 points to consider at any point during the sweep. All we need to do is output the extremal points at each “event” (e.g. highest and lowest).

What about the total number of “events” that can happen as we sweep from left to right? The next event can be either the rightmost endpoint of one of the polygon edges stabbed (4 possibilities) or it can be the intersection of two edges (4 possibilities). Thus, we encounter $O(n)$ events; each event can be processed in $O(1)$ time (e.g. find the upper or lower point to output); and the next event can be computed in $O(1)$ time. Hence, total time is $O(n)$.

3 Envelopes and Duality

3.1 Lower Envelope Problem

We are given $n$ lines $\{\ell_1, \ldots, \ell_n\}$. The lower envelope is the intersection of their lower halfplanes. The upper envelope is the intersection of their upper halfplanes.

3.2 Duality Definition

- Given any point $p = (a, b)$, let $p^*$ be the line $ax - b$.
- Given any line $\ell = ax + b$, let $\ell^*$ be the point $(a, -b)$

Primal and Dual planes

- **Primal plane**: Original $(x,y)$ plane
- **Dual plane**: The new $(a,b)$ plane
3.3 Duality Properties

- **Self-Inverse:** $p^{**} = p$

- **Order Reversing:** $p$ is above/on/below $\ell$ in the primal plane iff $p^*$ is below/on/above $\ell^*$ in the dual plane.

- **Preserves Vertical Distances:** The vertical distance between $p$ and $\ell$ equals the vertical distance between $p^*$ and $\ell^*$

- **Intersection preservation:** Lines $\ell_1$ and $\ell_2$ intersect at point $p$ in the primal plane iff the dual line $p^*$ passes through points $\ell_1^*$ and $\ell_2^*$

- **Colinearity/Coincidence:** Three points are collinear in the primal plane iff their dual lines intersect at the same point.

**Order Reversal Property**

To show point $p$ is above line $\ell$ iff line $p^*$ is below point $\ell^*$

- $p = (q, r)$ is above the line $\ell = sx + t \iff r \geq sq + t$

- $r \geq sq + t \iff qs - r \leq t$

- $qs - r \leq t \iff$ line $p^* = qx - r$ is below point $\ell^* = (s, t)$ is below the

3.4 Convex Hulls and Envelopes

**Lemma:** Let $P$ be a set of points in the plane. Then the ccw order of the upper convex hull of $P$ equals the left to right order of the lines on the lower envelope of the dual $P^*$. Symmetrically the order in the lower convex hull of $P$ is the left-to-right order of lines in the upper envelope.

**Proof:** Assume no three points are co-linear. Consider two consecutive points $q$ and $r$ on the upper hull and let $\ell$ be the line through these two points. By definition of the hull, $\ell$ is above all points in $P$. Thus, by the order reversing property, all dual lines of $P^*$ pass above point $\ell^*$. This means that point $\ell^*$ lies on the lower envelope. Next, consider the dual lines $q^*$ and $r^*$. By the incidence property, the dual point $\ell^*$ is the intersection of these two lines. (By general position, we assume the two points have different $x$-coordinates and so $q^*$ and $r^*$ have different slopes). Thus, the edges in the lower-envelope associated with the point $\ell^*$ are determined by the lines $q^*$ and $r^*$.
It remains only to show that the points in the lower envelope are ordered left-to-right. Note that as we move ccw along the upper hull (i.e. right-to-left), the slopes of the edges increase. Since the slope of the line in the primal plane determines the $x$-coordinate of the dual point, this means that we visit the lower envelope from left-to-right.

The proof for the upper envelope follows by symmetry.

Note that we can perform half-plane intersection via computing the upper and lower envelope problems as follows. Given a set of half-planes to intersect, first find the upper envelope of the half-planes that are open to the top, then find the lower envelope of the half-planes that are open to the bottom. Then merge these two envelopes (in $O(1)$ time), by computing the leftmost and rightmost intersection points.

### 3.5 Duality: A geometric View

Let $U$ be the parabola $y = x^2/2$. Consider some point $(p_x,p_y)$ on $U$. Note that $p^* = p_x x - p_y$.

Then:

- $p^*$ has same slope as the tangent line, since the derivative of $U$ at $p$ is $p_x$.
- $p^*$ has same $y$-intercept as tangent line, since the tangent line intersects the $y$-axis at $(0,p_x^2/2)$.
  - the tangent line equation is: $y = p_x (x) - b$, which implies that $b = p_y^2 - p_y = p_x^2 - p_x^2/2 = p_x^2/2$.
Thus for $p$ on $\mathcal{U}$, $p^*$ is exactly the tangent line at $p$. What about for any point $q = (q_x, q_y)$ not necessarily on $\mathcal{U}$?

- $q^*$ has the same slope as the tangent line at $q_x$.
- $q^*$ intersects the point $(q_x, q_y - (q_y - q_x^2 / 2))$.

### 3.6 Some Observations

- Duality can be applied to objects other than lines and points. For example, we can take the dual of a line segment.
- The dual of a segment is a double wedge. See Figure 5.
- Q: What line would dualize to a point in the right side of the figure?

### 4 Arrangements

An application of duality. In a set of $n$ points, are there 3 that are on a line. Naive algorithm takes $O(n^3)$ time. Faster algorithm uses duality and arrangements. Detecting 3 points on a line dualizes to detecting 3 lines intersecting at a point!

### 4.1 Arrangement Definition and Complexity

An arrangement of lines consists of faces, edges and vertices that are created via a set of lines in the plane.

Combinatorial complexity.
Lemma: An arrangement in the plane has

- \( \binom{n}{2} \) vertices
- \( n^2 \) edges
- \( \binom{n}{2} + n + 1 \) faces

Proof: First note that the number of vertices equals the number of intersection points of lines. Assuming general position, this is \( \binom{n}{2} \) vertices.

Next we prove the bound on the edges by induction. Base case is trivial. Inductive step: When we add a new line to an arrangement of \( n - 1 \) lines, we start with \( (n - 1)^2 \) edges by the IH. Then the new line splits exactly one edge from each of the existing lines. Also, the new line is split into \( n \) edges by its intersection with the other lines. Thus, the total number of edges in the arrangement with the new line is \((n - 1)^2 + (n - 1) + n = n^2\).

The number of faces follows by Euler’s formula \( v - e + f = 2 \). \( \square \)

Note that we can represent the arrangement fully as:

- The vertex and edge graph as shown in Figure 6
- A pointer for each edge to the two faces that edge is in
- For each face, a linked list that maintains the edges of the face in CCW order

Thus for \( d = 2 \), the complexity of an arrangement is \( O(n^2) \). It turns out that, for all \( d \geq 2 \), the complexity of an arrangement is \( O(n^d) \).

4.2 Building an Arrangement

We build an arrangement by adding one line after another. Let \( L = \{\ell_1, \ldots, \ell_n\} \) denote the set of lines. We show that line \( \ell_i \) can be added in \( O(i) \) time. This gives a \( O(n^2) \) time algorithm for building the entire arrangement.

Suppose the first \( i - 1 \) lines have been inserted and consider the insertion of \( \ell_i \).

First, determine the leftmost unbounded face containing \( \ell_i \). Assume that at \( x = \infty \), the lines are sorted by increasing slope. Then can find in \( O(i) \) time where \( \ell_i \) falls in terms of slope. This determines the leftmost face containing \( \ell_i \).

Next, the line \( \ell_i \) cuts through a bunch of edges and faces that we need to split up. To determine which edges are cut, we “walk” the line through the arrangement from one face to the next. When we enter a face, we need to determine through which edge \( \ell_i \) exits that face. We do this in a brute-force manner by just walking the edges in ccw order till we find the edge at which \( \ell_i \) exits the face. We then jump to the face on the other side of that edge. See Figure 7(a).
Problem: Naively, we can say that we encounter $i - 1$ lines, and hence pass through $i$ faces. Since each face is bounded by at most $i$ lines, we take $O(i^2)$ time. But this is slower than what was claimed. To improve the analysis, we need the zone theorem.

4.3 The Zone Theorem

Given an arrangement $A(L)$ and a line $\ell$ not in $L$, the zone of $\ell$ in $A(L)$, denoted $Z_{A(\ell)}$, is the set of faces of the arrangement that are intersected by $\ell$. The Zone theorem below states that the complexity of the zone (i.e. number of edges of all faces in the zone) is $O(n)$. The proof below is from [1].

**Theorem:** Given an arrangement $A(L)$ of $n$ lines in the plane, and given any line $\ell$ in the plane, the total number of edges in all the faces that $\ell$ intersects is at most $6n$.

**Proof:** For simplicity, rotate the plane so that $\ell$ is horizontal. We call an edge a left bounding edge for the face lying to the left of it, and a right bounding edge for the face lying to the right (Figure 8(a)). Otherwise, call the edge right bounding. We show that there are at most $3n$ left bounding edges. A symmetrical argument works for right-bounding edges.

Base Case: $n = 1$, there is exactly one left bounding edge in $\ell$’s zone, and $1 \leq 3n = 3$.

Inductive Step: Consider an arrangement of $n$ lines and let $\ell_1$ be the line that intersect $\ell$ at the rightmost point. By the inductive hypothesis, there are at most $3(n-1)$ left-bounding edges in the zone for $\ell$ in the arrangement $A(L - \ell_1)$.

Consider the rightmost face intersected by $\ell$ and note that all edges of this face are left-bounding edges. Line $\ell_1$ intersects $\ell$ within this face and, by convexity, it intersects the boundary of this face in two edges, call them $e_a$ and $e_b$. The insertion of $\ell$ into this face creates a new left bounding edge along $\ell$, and also splits $e_a$ and $e_b$ into two new left bounding edges for a net increase of 3 edges (Figure 8(b)). Note that $\ell_1$ can not contribute any other left-bounding edges to the zone: to the left of $\ell_1$, the edges induced by $\ell_1$ can form right-bounding edges only; to the right of $\ell_1$, all other faces possibly touched by $r$ are shielded from $\ell$ by either the line supporting $e_a$ or the line supporting $e_b$.

Thus, the total number of left bounding edges on the zone of $\ell$ in $A(L)$ is at most $3(n-1) + 3 \leq 3n$. $\square$
We claim that it is an easy matter to compute the level of each vertex of the arrangement (e.g., by definition of level is exactly one greater than our text’s definition.)

In order to count the number of lines lying above/below a vertex of the arrangement, it will be useful to count the number of each vertex in the arrangement and the number of lines that lie below it. If we know the number of lines passing through two points of the primal plane to a point and compute the discrepancy for the corresponding primal line.

Call the lines satisfying (i) as \( \ell \) and the left bounding edges created by \( \ell \).

5 Some Applications

5.1 Three Points on a line?

Given a set \( P \) of \( n \) points, in \( O(n^2) \), we can determne there are 3 points on a line (i.e. are the points in general position?).

Algorithm:

1. Create the arrangement for the lines in \( P^* \)
2. If 3 lines in \( P^* \) intersect at the same vertex, then the points associated with those 3 lines in the primal plane are co-linear.

This takes just \( O(n^2) \) time to compute the arrangement!

5.1.1 Levels of an Arrangement

We say that a point is at level \( k \) (denoted \( L_k \)) in an arrangement if there are at most \( k - 1 \) lines above the point and at most \( n - k \) lines below. See Figure 9(a). The upper envelope of the lines is level 1 of the arrangement, and the bottom envelope is level \( n \). Note that each point in the arrangement is generally on two levels, and that the levels are piecewise linear.

We can compute the levels of an arrangement in \( O(n^2) \) time as follows. First we sort all the lines via slope. Next, for each line, \( \ell \) in the arrangement, we first compute the level of the leftmost vertex \( (x = -\infty) \) of \( \ell \) in this sorted order in \( O(n) \) time. Next we walk along \( \ell \) from left to right using the edge lists. We maintain the level as we walk. The level only changes at a vertex and its change can be computed by inspecting edges incident to that vertex.

Alternatively, we can use plane-sweep from left to right to do this in \( O(n^2 \log n) \) time (keeping “events” in a priority queue - hence the \( \log n \) factor). Whenever, we come to the next vertex, we swap the level numbers associated with the two lines at the intersection. See Figure 9(b).

5.2 Ham-sandwich algorithm

We now discuss a discrete version of the celebrated “ham-sandwich” theorem.

\[ \text{The log } n \text{ factor can be eliminated via topological plane sweep.} \]
Given: $R$ and $B$ are two finite sets of points.

Goal: Output a line that bisects both $R$ and $B$. That is in either open half-plane defined by the line, there are no more than $|R|/2$ points from $R$ and no more than $|B|/2$ points from $B$.

Theorem: We can always find such a line in $O(n^2)$ time.

Proof: Assume $|R|$ and $|B|$ are both odd. If either set is not, remove an arbitrary point from it. Then, a bisector of the remaining sets will also bisect the original sets.

Now create an arrangement of both $A(R^*)$ and $A(B^*)$, and compute the levels for both the red and blue points. Let the median level for the red points be the level $[(|R| + 1)/2$, and note that it defines bisecting lines for $R$. The leftmost and rightmost segment of this median level are defined by the same line, $\ell_r$, since slope alone determines the ordering of the lines at $\pm\infty$. Similarly, there is a line $\ell_b$ that determines the leftmost and rightmost segment of the median level for $A(B^*)$.

Note that $\ell_r$ and $\ell_b$ have different slopes since no two point in $R \cup B$ have the same $x$ coordinate. Thus, since the median levels for $A(R^*)$ and $A(B^*)$ are piecewise linear functions, these median levels must also intersect somewhere in $A(R^* \cup B^*)$ (see Figure 10(right)). In the primal plane, this intersection point represents a half-plane that bisects $R$ and $B$. \hfill \square

5.3 Red Blue Matching

Given $n$ red and $n$ blue points in the plane, we can pair them up using non-intersecting line segments!

Proof (by recursive algorithm).

1. If $|P| = 2$, match up the two points with a line segment

2. If $|P| > 2$
   
   (a) Compute a ham-sandwich bisecting cut that bisects
   
   (b) If $|P|$ is odd, match the red and blue point on the bisecting line
   
   (c) If $|P|$ is even, rotate or translate the cutting line slightly so that (1) no points are on the line and (2) the line is still a bisecting cut

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\footnote{By general position assumption - can rotate the plane slightly to prevent this if it’s true.}
(d) Recursively compute a matching for both $P_1$ and $P_2$, where $P_1$ and $P_2$ are the points on either side of the cut.

5.4 Halfplane Discrepancy

Discrepancy is a problem closely related to sampling, and it measures how well a discrete sample can approximate a continuous structure. Initially, it was important in computer graphics for determining how well a collection of random rays would be spread over a continuous surface. But it has new applications in big data (finding how well a small sample represents a larger population), and also probability and randomized algorithms (e.g. discretizing continuous probability distributions).

We are given a set, $P$ of $n$ points in the unit square $U$, and we want for any halfplane $h$ that the fraction of points lying in the halfplane is close to the area of $h \cap U$. If we define $\mu(h)$ to be the area of $h \cap U$ and $\mu_P(h) = |P \cap h|/|P|$, then we can define the discrepancy for a point set $P$ and a halfplane $h$ as:

$$\Delta_P(h) = |\mu_h - \mu_P(h)|$$

We define the halfplane discrepancy of $P$ as supremum (least upperbound) of this quantity over all $h$:

$$\Delta(P) = \sup_h |\mu_h - \mu_P(h)|.$$

The Discrepancy Problem

**Given:** A set of $n$ points $P$ lying in the unit square.

**Goal:** Compute $\Delta(P)$ and return a halfplane that achieves this discrepancy.

5.4.1 The Solution

The following lemma helps us compute $\Delta(P)$.

**Lemma:** Let $h$ be the halfplane that maximizes discrepancy with respect to $P$. Let $\ell$ denote the line that bounds $h$. Then either (1) $\ell$ passes through one point of $P$ and this point is the midpoint of the line segment $\ell \cap U$; or (2) $\ell$ passes through two points of $P$. 

![Figure 11. Computing Discrepancy. (4)](image-url)
Proof: Consider a halfplane induced by a line $\ell$ that does not intersect any points. Then the discrepancy can be increased by either moving $\ell$ up or down (whichever direction increases discrepancy) until it hits a point. (See Figure 11 (a)). Thus $\ell$ must intersect at least one point.

Now consider the case where $\ell$ intersects just one point, but that this point does not form the midpoint of the line segment $\ell \cap U$. In this case, there are two line segments $r_1$ and $r_2$ of different length that are induced by the point (See Figure 11 (b)).

Assume $r_1 < r_2$, and consider the rotation of $h$ about $p$ by a small angle $\theta$. Then the asymptotic approximation to the gain due to the area on the right is $r_1^2 \theta/2$, since this triangle can be approximated by an sector of angle $\theta$ in a circle of radius $r_1$. The loss due to the area on the left is similarly asymptotically $r_2^2 \theta/2$.

Thus this rotation will decrease the area of the region lying below $h$. Similarly a rotation in the opposite direction will increase the area of the region lying below $h$. Hence, it is again possible to increase the discrepancy via rotation in the appropriate direction. \hfill \square

Given this lemma, we can solve the problem in $O(n^2)$ time as follows. Let a type 1 line be one that passes through one point of $P$ and this point is the midpoint of the line segment $\ell \cap U$. Let a type 2 line be one that passes through two points of $P$. Note that there are only $O(n)$ type 1 lines, so we can easily compute the discrepancy of the halfplanes induced by these lines in $O(n^2)$ time.

For the type 2 lines, we use arrangements and levels. In particular, we know that any type 2 line will be a vertex in the arrangement in the dual of $P^*$, i.e. in $A(P^*)$. For each such vertex $p \in A(P^*)$, we can compute the area of the two halfplanes induced by the line $p^*$ in the bounding box $U$ in constant time. Thus we can compute $\mu_h$ in constant time per type-2 halfplane $h$.

Moreover, we can compute the levels of all the points in $A(P^*)$ in $O(n^2)$ time as described in the Section 5.1.1. This allows us to compute in constant time the value $\mu_P(h)$ in constant time per type-2 halfplane $h$.

Since the total number of type-2 halfplanes is $O(n^2)$ and we can compute the discrepancy induced by each in constant time, we can also find the maximum discrepancy of these type-2 halfplanes in $O(n^2)$.

6 Duality and Arrangements in Higher Dimensions

In $\mathbb{R}^d$, identify the $y$ axis with the $d$-th coordinate. Then a point can be written as $p = (x_1, \ldots, x_{d-1}, y)$ and a $d-1$ dimensional hyperplane, $h$ is $y = \sum_{i=1}^{d-1} a_i x_i - b$. The dual of the hyperplane is the point $h^* = (a_1, \ldots, a_{d-1}, b)$ and the dual of the point is the hyperplane $b = \sum_{i=1}^{d-1} x_i a_i - y$. All the properties general to point/hyperplane relationships under the assumption that the $y(b)$ axis is “vertical”.

An arrangement of $d$ dimensional hyperplanes has combinatorial complexity $O(n^d)$.

References


\footnote{Technically, we can say that as $\theta$ goes to 0, the area of the triangle is $r_1^2 \theta/2 + o(r_1)$, since the error in approximating the shape as a sector angle versus a triangle is $o(r_1)$}