

Figure 1. Voronoi Diagram of point set P , denoted $\text{Vor}(P)$

Note: These lecture notes are based on the textbook “Computational Geometry” by Berg et al. and lecture notes from [3]

1 Voronoi Diagrams

A Voronoi diagram encodes proximity information, i.e. what is close to what. Let P be a set of points in the plane (or more generally in \mathbb{R}^d), and for any two points p, q , let $|p - q| = \left(\sum_{i=1}^d (p_i - q_i)^2\right)^{1/2}$ be the Euclidean distance between p and q . Define $\mathcal{V}(p)$, the *Voronoi cell* for p to be the set of points q in the plane that are closer to p than to any other $p' \in P$. More formally:

$$\mathcal{V}(p) = \{q \in \mathbb{R}^d : |q - p| < |q - p'|, \forall p' \in P - p\}$$

The union of the closures of the Voronoi cells defines a cell complex called the *Voronoi diagram* of P , denoted $\text{Vor}(P)$.

1.1 Convex Polyhedra

The cells of the Voronoi diagram are possibly unbounded convex polyhedra. To see this, fix two points $p, p' \in P$ and note that the set of points closer to p than p' is equal to an open halfplane, whose bounding hyperplane is the perpendicular bisector of the line segment $\overline{pp'}$. Denote this halfplane $h(p, p')$. Now note that

$$\mathcal{V}(p) = \bigcap_{q \in P - p} h(p, q)$$

Since the intersection of convex spaces is convex and the intersection of polyhedra are polyhedra, $\mathcal{V}(p)$ is a (possibly unbounded) convex polyhedra.

1.2 Applications

Imagine the sites P are post offices and we want to compute the regions that are best served by each post office, when cost to visit a site is a linear function of the Euclidean distance to that site.¹ The Voronoi diagram $\text{Vor}(P)$ exactly delineates those regions. Moreover, it's not hard to generalize the concept of the Voronoi diagram to the case where costs to sites may be determined by *different*

¹this also works with e.g. supermarkets

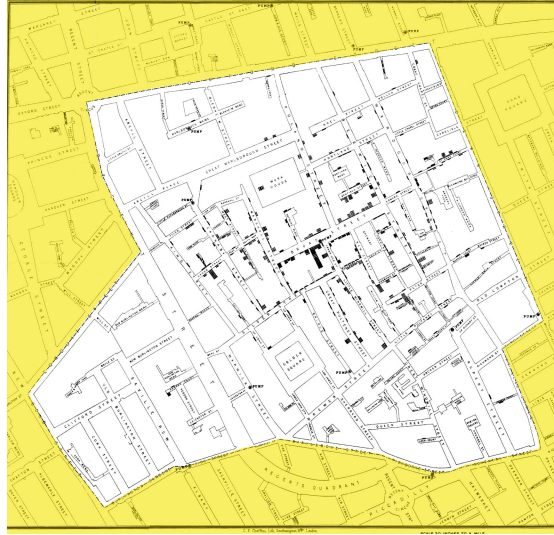


Figure 2. Approximate Voronoi Cell of the water pump responsible for the 1854 cholera outbreak in London. Bars represent cholera deaths.

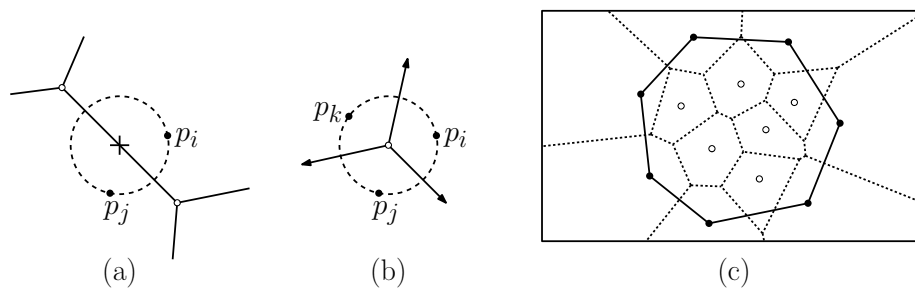


Figure 3. Properties of Voronoi Diagram

linear functions of the Euclidean distance. Voronoi diagrams have been used by “*anthropologists to describe regions of influence of different cultures; by crystallographers to explain the structure of certain crystals and metals; by ecologists to study competition between plants; and by economists to model markets in the U.S. economy.*” [1] They were even used by John Snow to isolate the pump responsible for the 1854 London Cholera outbreak! (See Figure 2.)

Nearest Neighbor Queries. After construction and processing of a Voronoi diagram of point set P , we can answer nearest neighbor queries in $O(\log n)$ time. The easiest way to do this is to first construct vertical slabs that are delineated by the vertices in the Voronoi diagram. Sort these slabs by x -coordinate so that in logarithmic time, it is possible to find which slab a new point falls in. Next, sort the line segments that intersect each slab along y -coordinates, so that given the slab that a point falls in, in logarithmic time it is possible to find which cell of the Voronoi diagram that point falls in. Since the number of slabs and line segments are polynomial, lookups take logarithmic time.²

1.3 Properties

Theorem 1. *Let P be a set of n sites in the plane. If all sites are colinear then $\text{Vor}(P)$ consists of $n - 1$ parallel lines. Otherwise $\text{Vor}(P)$ is connected and its edges are all line segments or half lines.*

Proof: The first part holds trivially, so assume not all sites are colinear.

We first show that the edges of $\text{Vor}(P)$ are either line segments or half-lines. Assume by way of contradiction that there is an edge e that is a full line. Let e be on the boundary of the Voronoi cells $\mathcal{V}(p_1)$ and $\mathcal{V}(p_2)$ for $p_1, p_2 \in P$. Let p_3 be a site that is not collinear with p_1 and p_2 . The bisector of p_2 and p_3 is not parallel to e and so it intersects e . But then the part of e that lies in the interior of $h(p_3, p_2)$ cannot be on the boundary of $\mathcal{V}(p_2)$, since it is closer to p_3 than p_2 , a contradiction.

Now assume that $\text{Vor}(P)$ is not connected. Then, there would be a Voronoi cell $\mathcal{V}(p_1)$ that splits the plane in two. Since Voronoi cells are convex, $\mathcal{V}(p_1)$ would be a vertical strip bounded by two parallel lines. But we just proved that no edge of the Voronoi diagram can be a line, a contradiction. \square

Theorem 2. *The following holds for $\text{Vor}(P)$ of a set of points P .*

- **Voronoi vertices.** *A point q is a vertex of $\text{Vor}(P)$ iff its largest empty circle, called $C_P(q)$, contains three or more sites on its boundary.*
- **Voronoi edges.** *The bisector between sites p_1 and p_2 defines an edge of $\text{Vor}(P)$ iff there is a point q on the bisector such that $C_P(q)$ contains both p_1 and p_2 on its boundary but no other sites. (Then all such points q are on the Voronoi edge)*

Proof: Voronoi vertices: Suppose there is a point q such that $C_P(q)$ contains three or more sites on its boundary. Let p_1, \dots, p_ℓ be these sites. Since the interior of $C_P(q)$ is empty, point q must be on the boundary of each of $\mathcal{V}(p_1), \dots, \mathcal{V}(p_\ell)$. Hence, point q is a vertex of $\text{Vor}(P)$.

On the other hand, assume point q is a vertex of $\text{Vor}(P)$. We first show that q is incident to at least 3 edges. To see this, assume that q was incident to only 2 edges, supported by the half-planes $h(p_1, p_2)$ and $h(p_1, p_3)$. Then, note that at point q , sites p_2 and p_3 must also be equidistant. Thus q also touches the edge supported by the half-plane $h(p_2, p_3)$.

Hence, q is incident to at least three edges, and thus incident to at least three Voronoi cells $\mathcal{V}(p_1), \mathcal{V}(p_2), \mathcal{V}(p_3)$, for $p_1, p_2, p_3 \in P$. Voronoi vertex q is equidistant to p_1, p_2, p_3 and there can not be a site closer to q . Hence, $C_P(q)$ is an empty circle containing three or more sites on its boundary. (See Figure 3 (b))

Voronoi edges: Suppose there is a point q on the bisector between sites p_1 and p_2 such that $C_P(q)$ contains p_1 and p_2 on its boundary but no other sites. Then, $\text{dist}(q, p_1) = \text{dist}(q, p_2) < \text{dist}(q, p_x)$ for any other site $p_x \in P - \{p_1, p_2\}$. Hence q lies on an edge of $\text{Vor}(P)$ that is defined by the bisector of p_1 and p_2 .

On the other hand, let the bisector of p_1 and p_2 define a Voronoi edge. Then the largest circle of any point q on this edge must contain p_1 and p_2 on its boundary and no other sites. (See Figure 3 (a)) \square

Theorem 3. *The number of vertices, faces and edges in $\text{Vor}(P)$ are all $O(n)$.*

²Note that there are more space-efficient ways to do point lookup using something called trapezoidal maps.

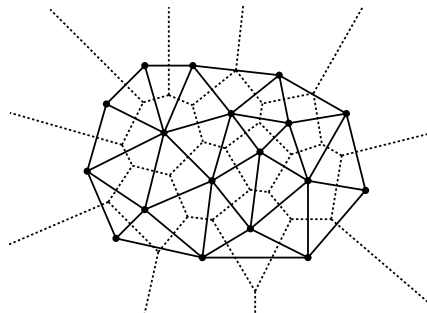


Figure 4. Delaunay triangulation of a set of points (solid lines) and the Voronoi diagram (dashed lines).

Proof: Create a planar graph from the $\text{Vor}(P)$ by adding an extra vertex v_∞ at “infinity”, and connecting all half-infinite edges of $\text{Vor}(P)$ to this vertex.

Recall that Euler’s formula says that $(V + 1) - E + F = 2$ (we add one to V because of v_∞) If the number of sites is n (i.e. $|P| = n$) then the number of faces in the Voronoi diagram is n . Recall that the sum of all degrees is twice the number of edges (handshaking lemma), and every vertex has degree at least 3. Thus, $2E \geq 3(V + 1)$. So we have

$$\begin{aligned} V + 1 &= (2 - n) + E \\ &\geq (2 - n) + (3/2)(V + 1) \end{aligned}$$

From whence, we get $V + 1 \leq 2(n - 2)$. Using this to bound E , we get that $E \leq 3n - 6$. □

1.3.1 Additional Properties

Degree. Three points in the plane define a unique circle. If we make a general position assumption that no four sites are cocircular, then each vertex of the Voronoi diagram is incident to 3 edges. In \mathbb{R}^d , the vertex is defined by $d + 1$ points in general position, and the hypersphere centered at the vertex passing through these sites is empty.

Convex Hull. A cell of the Voronoi diagram is unbounded iff the corresponding site is on the convex hull of P . To see this, note that a site is on the convex hull iff it is the closest site from some point at infinity. (See Figure 3 (c).) Thus, given a Voronoi diagram it is easy to compute the convex hull. Later we’ll see the reverse is also true.

1.4 Computing the Voronoi Diagram

How can we efficiently compute a Voronoi diagram? There are several direct algorithms, and also we’ll see how we can use a (higher-dimensional) convex hull algorithm to compute it! To do this, we show connections between: Voronoi Diagrams and Delaunay graphs (Section 2); Delaunay and Convex Hulls (Section 3); and Voronoi and Convex Hulls (Section 4). Finally, we’ll show a sweep-line algorithm, Fortune’s algorithm, to directly compute the Voronoi diagram (Section 5). Fortune’s algorithm has the benefit of being able to directly handle weighted Voronoi diagram problems.

2 Delaunay Triangulation

The Voronoi diagram is a planar subdivision that subdivides the plane into convex (possibly open) polyhedra.

The *Delaunay graph* is the dual³ of the Voronoi graph defined as follows (See Figure 4).

- For each face of the Voronoi graph, we create a vertex corresponding to the face's site.
- For each edge of the Voronoi graph lying between sites p and q , we create an edge in the dual connecting the two vertices associated with these sites.
- Thus, each vertex of the Voronoi graph corresponds to a face in the dual.

If no four points are on the same circle (i.e. general position). Then all vertices of the Voronoi have degree 3 and so all faces of Delaunay are triangles. This is why the Delaunay graph is generally called the *Delaunay triangulation*.

2.1 Properties of Delaunay

Theorem 4. *The Delaunay triangulation of n points P in the plane, forms a planar graph, where the number of faces, edges and vertices are all $O(n)$.*

Proof: This follows from Theorems 1 and 3 and the duality property between Delaunay graphs and Voronoi graphs. \square

In 3-space, the number of tetrahedra in the Delaunay triangulation can range from $O(n)$ to $O(n^2)$. In dimension d , the number of simplices (d dimensional generalization of a triangle) can be $\Theta(n^{\lceil d/2 \rceil})$.

Theorem 5. *Let P be a set of points in the plane. Then*

1. *Three sites $p_1, p_2, p_3 \in P$ are vertices of the same face of the Delaunay graph of P iff the circle through p_1, p_2, p_3 contains no sites of P in its interior*
2. *Two sites $p_1, p_2 \in P$ form an edge of the Delaunay graph of P iff there is a circle that has p_1 and p_2 on its boundary and contains no other sites of P*

Proof: By Theorem 2 (1), we know that a point q is a vertex of $\text{Vor}(P)$ iff its largest empty circle $C_p(q)$ contains three or more sites on its boundary. Since Voronoi points become faces in the Delaunay triangulation, this translates to Property (1) of the theorem by duality.

By Theorem 2 (2), we also know that the bisector between sites p_1 and p_2 defines an edge of $\text{Vor}(P)$ iff there is a point q on the bisector such that there is a circle with p_1 and p_2 on its boundary but containing no other sites. Since Voronoi edges associated with bisectors between p_1 and p_2 become Delaunay edges connecting p_1 and p_2 , this translates to Property (2) of the theorem. \square

Theorem 6. *Three sites in P form a Delaunay triangle if and only if no other site of P lies in the closed circumcircle defined by the points.*

Proof: Consider three sites $p_1, p_2, p_3 \in P$. By Theorem 5 (1) they are the only sites on the same face of the Delaunay graph iff the closed circle through p_1, p_2, p_3 contains no other sites in P . If only these three sites are on the same face, then clearly all edges exist between them, and so they form a Delaunay triangle. \square

³This is another type of duality, different from the point/line duality of last lecture

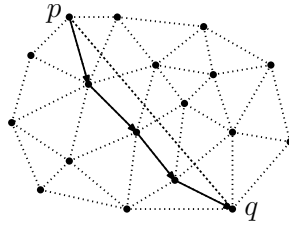


Figure 5. Spanner property of Delaunay triangulation.

2.2 Applications and Additional Properties

Spanner Properties. The length of the shortest path between two points through edges in the planar Delaunay triangulation is not too much longer than the Euclidean distance between these two points. In particular, the increase in distance by using only edges in the Delaunay triangulation is at most $4\pi\sqrt{3}/9 \approx 2.418$. This is proven in a paper by Keil and Gutwin.

Maximizing Angles. Among all triangulations, the Delaunay maximizes the minimum angle. This is useful because in many applications, we want to avoid skinny triangles, for better interpolation. In fact a stronger statement holds. Among all triangles with the smallest angle, the Delaunay triangulation maximizes the second smallest angle, and so on.

Convex hull. Similar to the Voronoi diagram, the boundary of the exterior face of the Delaunay triangulation is the boundary of the convex hull of the point set.

3 Delaunay to Convex Hull

We'll now show a neat connection between solving Delaunay in 2 dimensions and Convex Hull in 3 dimensions.

Let Ψ be the paraboloid $z = x^2 + y^2$. For any point $p = (p_x, p_y) \in \mathbb{R}^2$, define the *vertical projection* (or *lifted image*) of p onto Ψ to be point $p^\uparrow = (p_x, p_y, p_x^2 + p_y^2)$ in \mathbb{R}^3 . Given a set of points in the plane, P , let P^\uparrow be the projection of every point in P onto Ψ . Let the *lower convex hull* of P^\uparrow be the part of the convex hull visible to an observer at $z = -\infty$.

We will show the Delaunay condition (from Theorem 6), and the lower convex hull condition are equivalent.

Delaunay condition. Three sites in P form a Delaunay triangle if and only if no other site of P lies within the circumcircle defined by the sites.

Lower Convex hull condition. Three points $p^\uparrow, q^\uparrow, r^\uparrow \in P^\uparrow$ form a face of the lower convex hull of P^\uparrow if and only if no other point of P^\uparrow lies below the plane passing through $p^\uparrow, q^\uparrow, r^\uparrow$.

To show this connection, we want to show a connection between the emptiness of circumcircles in the plane and the emptiness of lower halfspaces in 3-space. The following lemma proves this. It is illustrated in Figure 6.

Lemma 1. Consider four distinct points p, q, r and s in the plane. The point s lies within the circumcircle of Δpqr if and only if s^\uparrow lies beneath the plane passing through p^\uparrow, q^\uparrow and r^\uparrow .

Proof: First consider an arbitrary plane in \mathbb{R}^3 that is tangent to Ψ above some point (a, b) in the xy -plane. To get the equation for this plane, we take derivatives of ψ at the tangent point. For the

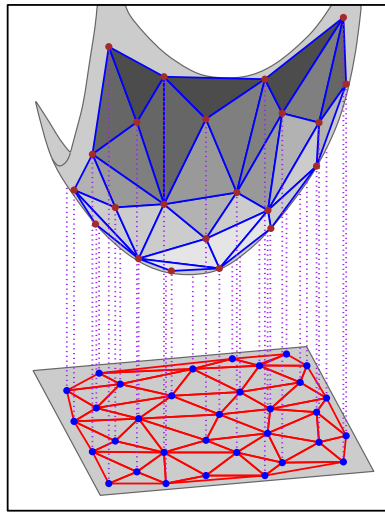


Figure 6. The Delaunay triangulation and Convex Hull.

equation $z = x^2 + y^2$, $\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 2y$. At the tangent point $(a, b, a^2 + b^2)$, these evaluate to $2a$ and $2b$, so the tangent plane passing through (a, b) has the form

$$z = 2ax + 2by + \gamma$$

[Jared: In class: solve for γ , shift plane upward by some amount h^2 , and show that the projection of intersection onto the x, y -plane is a circle. To solve for γ use the fact that the plane passes through $(a, b, a^2 + b^2)$]

To solve for γ , we use the fact that the plane passes through $(a, b, a^2 + b^2)$ to get that

$$a^2 + b^2 = 2a \cdot a + 2b \cdot b + \gamma$$

or $\gamma = -(a^2 + b^2)$. Thus the plane equation is

$$z = 2ax + 2by - (a^2 + b^2) \tag{1}$$

Now if we shift the plane upwards by some positive amount h^2 , we get the plane

$$z = 2ax + 2by - (a^2 + b^2) + h^2.$$

The intersection of this with Ψ is

$$x^2 + y^2 = 2ax + 2by - (a^2 + b^2) + h^2$$

which after rearrangement is:

$$(x - a)^2 + (y - b)^2 = h^2$$

So we've shown that the intersection of a plane with ψ is a space curve that when projected back onto the x, y plane gives a circle, whose radius is the square root of the distance by which the plane has been translated upwards. Thus, the intersection of an arbitrary lower halfspace with Ψ , when projected onto the x, y -plane is the interior of a circle!

Returning to the lemma, if we project points p, q and r from the xy -plane onto Ψ , the points $p^\uparrow, q^\uparrow, r^\uparrow$ define a plane. Since $p^\uparrow, q^\uparrow, r^\uparrow$ are in the intersection of this plane and Ψ , the original

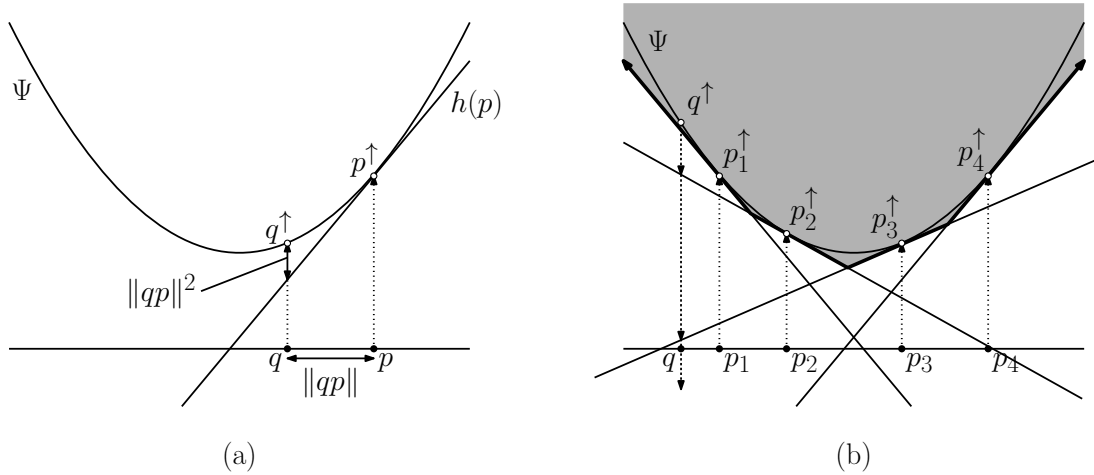


Figure 8. (a) Vertical distance from q^\uparrow to $h(p)$ equals $\|qp\|^2$ (i.e. $|q - p|^2$); (b) Thus, a vertical ray from q^\uparrow intersects the planes in $H(P)$ in the same order as distances from q to points in P in the xy -plane.

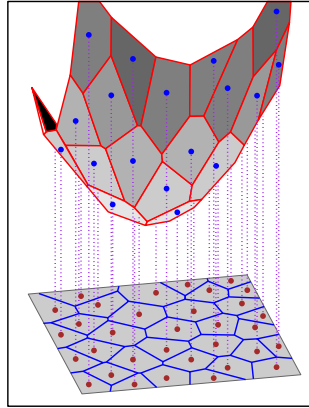


Figure 9. The Upper Envelope of hyperplanes tangent to P^\uparrow equals the Voronoi diagram when projected onto the xy -plane

Lemma 3. Given a set of points $P = \{p_1, \dots, p_n\}$ in the xy -plane, let $H(P) = \{h(p) : p \in P\}$. Then for any point q in the xy -plane, a vertical ray directed downwards from q^\uparrow intersects the planes of $H(p)$ in the same order as the distances of q from the points in P .

Proof: This follows immediately from Lemma 2. (Figure 8(b) illustrates the lemma.) \square

Theorem 8. Given a set P of points in the xy -plane, let $U(P)$ be the upper envelope of the tangent hyperplanes passing through each point p^\uparrow for $p \in P$. Then the Voronoi diagram of P is equal to the vertical projection on the xy -plane of the boundary complex of $U(P)$

Proof: Consider the upper envelope $U(P)$ of $H(P)$. This is an unbounded convex polytope, whose vertical projection covers the entire xy -plane. Now label every face of this polytope with the point $p \in P$ whose plane $h(p)$ defines the face. Then, by Lemma 3, the site p is closest to every point in the vertical projection of this face onto the plane. Thus, when $U(P)$ is projected on the xy -plane, it exactly gives the Voronoi diagram of P . (Figure 9). \square

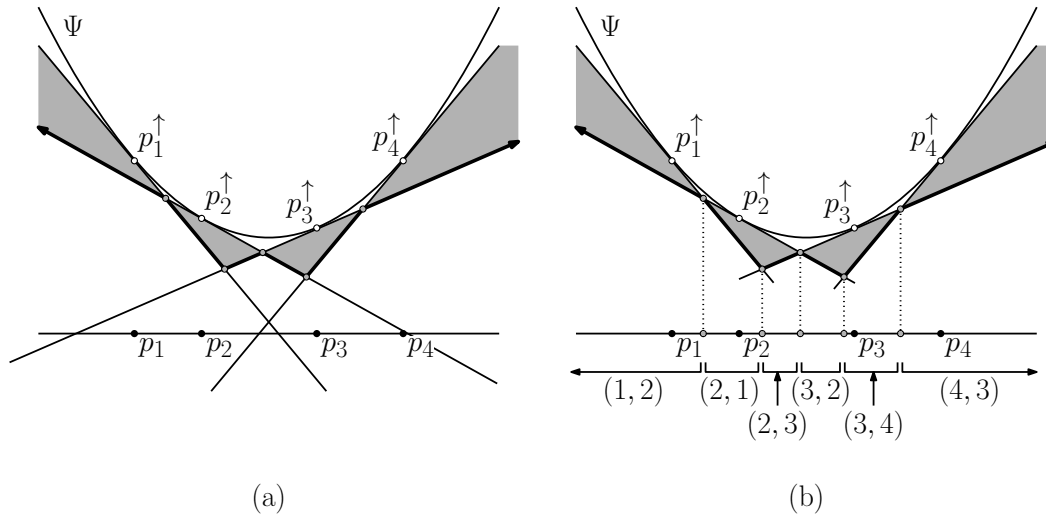


Figure 10. (a) Planes in $H(P)$ and the 2-level of their arrangement in bold; (b) The projection of the 2-level of this arrangement back onto the xy-plane. Note that this gives the order-2 Voronoi diagram of P .

4.1 Higher-Order Voronoi Diagrams and Arrangements

An *order- k* Voronoi diagram is a subdivision of the plane into regions, each associated with a subset of k sites that are the k nearest neighbors of any point in the region. For example, when $k = 2$, each cell of an order 2 Voronoi diagram is associated with two sites (p_i, p_j) , and the cell contains all points whose 2 closest sites are p_i and p_j . We next show that all order k Voronoi diagrams can be generated via projections onto Ψ .

Lemma 4. Let P be a set of points in the xy-plane, and let $H(P) = \{h(p) : p \in P\}$ be the set of hyperplanes defined above. Let A be an arrangement of these hyperplanes and let $\mathcal{L}_k(A)$ be the k -th level of the arrangement of A . Then the vertical projection of $\mathcal{L}_k(A)$ onto the xy-plane equals the order k Voronoi diagram of P .

Proof: Recall that in 2 dimensions, for any arrangement, A and any k , $1 \leq k \leq n$, the k -th level of the arrangement, $\mathcal{L}_k(A)$, consists of the line segments that have exactly k lines lying on or above them. Similarly, in 3 dimensions, for any arrangement A , and any k , $1 \leq k \leq n$, the k -th level of the arrangement, $\mathcal{L}_k(A)$, consists of the faces of the arrangement that have exactly k hyperplanes lying on or above them. By Lemma 3, the level k of the arrangement of $H(P)$, when projected vertically onto the xy-plane is exactly the order- k Voronoi diagram. (See Figure 10.) \square

Note that as shown in Figure 10, the projection actually gives a refinement of the order-2 Voronoi diagram because it distinguishes between e.g. the cells $(1, 2)$ and $(2, 1)$, depending on which of the two sites is closer.

Also note that the lower envelope of $H(P)$ is the order- n Voronoi diagram. This is called the farthest-point Voronoi diagram since each cell is defined by the farthest site from that cell.

5 Fortune's Algorithm for Voronoi Algorithm

The above projection onto a paraboloid approach gives us one way to compute Voronoi diagrams (projection plus half-plane intersection in 3d). Another interesting and direct way is to use For-

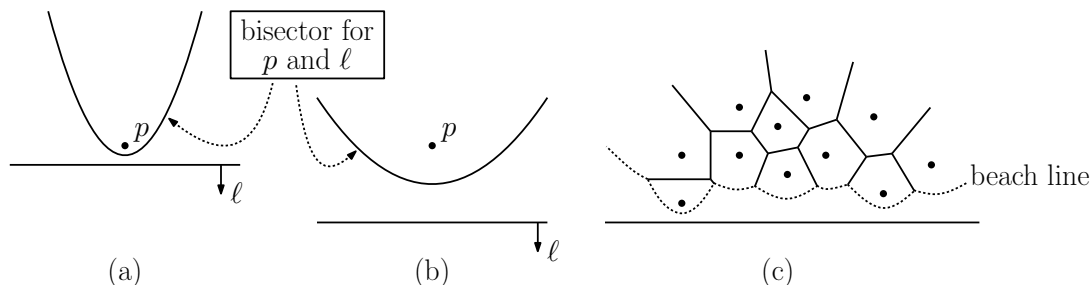


Figure 11. (a) and (b) parabolas defining the points closer to p than ℓ ; (c) the “beach line” is the lower envelope of the parabolas

tune’s algorithm⁴, which is another sweep line method. One benefit of Fortune’s algorithm is that it is generalizable in ways orthogonal to convex hull algorithms. In particular, it can more directly handle weighted Voronoi diagrams in 2D. It is a $O(n \log n)$ algorithm, which is optimal by a reduction from sorting. Clearly, by Duality, Fortune’s algorithm can also be used to efficiently compute the Delaunay triangulation.

Basic idea:

- A horizontal line sweeps the sites from top to bottom, we build the Voronoi diagram from top down.
- Set of parabolic arcs form a beach line that bounds the location of all points that are closer to the sites above than to the sweep line.

Also:

1. Points where two parabolas intersect are called “break points”
2. These break points trace out Voronoi edges

5.1 The Two Events

We maintain a sorted list (along x-axis) of parabolas that compose the lower envelope of the beach. Then there are two types of events that can occur as the sweep-line moves downward.

- **Site event.** The sweep-line intersects a new site in P (Figure 12). We do binary search to determine where to insert this new point in the beach line. The point initially forms a vertical line on the beach. This is the only way new points (and their parabolas) can be inserted into the beach line, and it increases the size by at most 2. Thus, our beach line list (top of figure) is $O(n)$.
- **Voronoi vertex event.** The sweep-line dips below the circumcircle of three adjacent points, p_i, p_j, p_k on the beach line (Figure 13). At this event, a Voronoi vertex is created in the Voronoi diagram, and the consecutive triple $p_i p_j p_k$, is replaced with $p_i p_k$ in the beach line list.

⁴Much of the discussion and figures in this section are from [2]

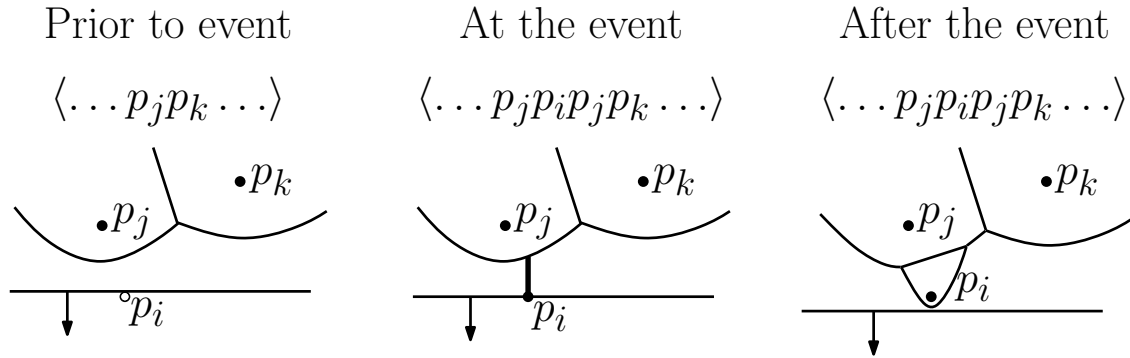


Figure 12. Site Event for site p_i . At the top is the sorted list of parabolas.

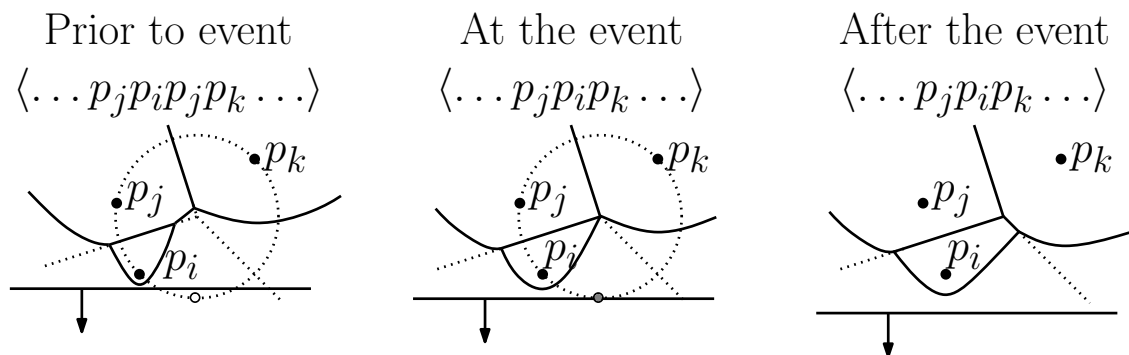


Figure 13. Voronoi vertex event

5.2 Sweep Line Algorithm

There are two key data structures we use in the algorithm.

Partial Voronoi Diagram. We can store this as usual as a planar graph with doubly-linked lists of edges around the faces. Since the graph is partial, will connect all currently incomplete edges in the Voronoi diagram to a special vertex at infinity.

Beach Line. This is the sorted list of sites that form the arcs in the beach line. We don't need to explicitly store the parabolas. The key search operation is finding the arc of the beach line that lies right above any newly intersected site.

How to find this? Between any pair of sites p_i and p_j there is break-point on the sweep-line, which is the center of a circle on the sweep line that has p_i and p_j on its borders. We can dynamically find this break-point in constant time, determine if the new site falls to the right or left of it, and then proceed with our binary search to find where the new site falls in the beach line in $O(\log n)$ time.

Event Queue. The event queue is a priority queue where we can insert and delete events, with keys equal to their y-coordinates. Initially, all sites are inserted into the priority queue. Additionally, for every consecutive triple $p_i p_j p_k$ currently on the beach line, if the bottom endpoint of their circumcircle is below the current sweep-line, we store a Voronoi vertex event in the priority queue

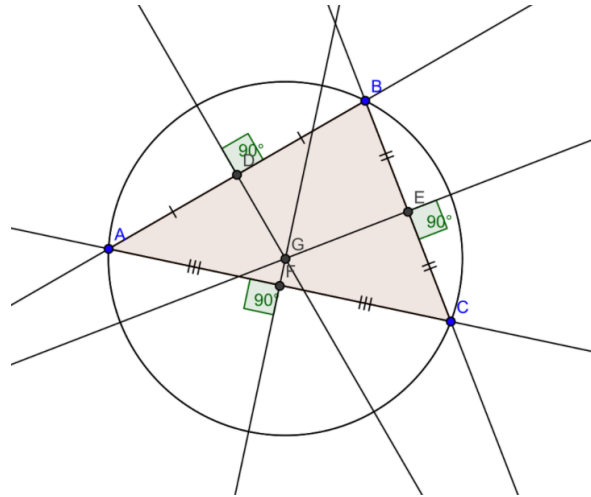


Figure 14. Circumcircle vertex event

with key equal to the smallest y value of this circle. We remove this event if $p_i p_j p_k$ ever cease to be consecutive.

Analysis. Each event takes $O(\log n)$ time to pull it out of the queue and process it. The size of all data structures are $O(n)$ and there are $O(n)$ total events processed. So total time is $O(n \log n)$

5.2.1 Weighted Sites

So how do we handle weighted sites? The main change is to update the method for adding Voronoi vertex events. In particular, we want to find a circle which intersects the “weighted” location of the three sites. To do this we draw the line segments for the triangle for the sites and then find the intersection of the perpendicular bisectors of these line segments. The intersection will be the center of the circle. Figure 14 shows the intersection of two of these lines when all 3 sites have the same weight.

When the weights are different, the perpendicular bisecting lines are located at a point in the line segment such that the ratio of the lengths of the two partitions of the sub-segments equals the ratio of the weights of the sites. Then the intersection of any two of these bisectors gives the center of the circle, and the radius is set so that the circle contains at least one site.

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