CS 561, Lecture 7

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Today's Outline

- Randomized Quicksort
- Sorting Lowerbound
- Bucket Sort
- Dictionary ADT

R-Partition

```
//PRE: A[p..r] is the array to be partitioned, p>=1 and r <= size
//     of A
//POST: Let A' be the array A after the function is run.  Then
//      A'[p..r] contains the same elements as A[p..r].  Further,
//      all elements in A'[p..res-1] are <= A[i], A'[res] = A[i],
//      and all elements in A'[res+1..r] are > A[i], where i is
//      a random number between $p$ and $r$.
R-Partition (A,p,r){
  i = Random(p,r);
  exchange A[r] and A[i];
  return Partition(A,p,r);
}
```


Randomized Quicksort

```
//PRE: A is the array to be sorted, p>=1,  and r is <= the size of A
//POST: A[p..r] is in sorted order
R-Quicksort (A,p,r){
  if (p<r){
    q = R-Partition (A,p,r);
    R-Quicksort (A,p,q-1);
    R-Quicksort (A,q+1,r);
}
```

Analysis

- R-Quicksort is a *randomized* algorithm
- The run time is a *random variable*
- We'd like to analyze the *expected* run time of R-Quicksort
- To do this, we first need to learn some basic probability theory.

Plan of Attack

"If you get hold of the head of a snake, the rest of it is mere rope" - Akan Proverb

- We will analyze the *total* number of comparisons made by quicksort
- We will let X be the total number of comparisons made by R-Quicksort
- We will write X as the sum of a bunch of indicator random variables
- We will use linearity of expectation to compute the expected value of X

Notation

- Let A be the array to be sorted
- Let z_i be the i-th smallest element in the array A
- Let $Z_{i,j} = \{z_i, z_{i+1}, \ldots, z_j\}$

Indicator Random Variables

- Let $X_{i,j}$ be 1 if z_i is compared with z_j and 0 otherwise
- Note that $X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}$
- Further note that

$$E(X) = E(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{i,j})$$

Questions

- Q1: So what is $E(X_{i,j})$?
- A1: It is $P(z_i \text{ is compared to } z_j)$
- Q2: What is $P(z_i \text{ is compared to } z_j)$?
- A2: It is:

  $P(\text{either } z_i \text{ or } z_j \text{ are the first elems in } Z_{i,j} \text{ chosen as pivots})$

- Why?
  - If no element in $Z_{i,j}$ has been chosen yet, no two elements in $Z_{i,j}$ have yet been compared, and all of $Z_{i,j}$ is in same list
  - If some element in $Z_{i,j}$ other than z_i or z_j is chosen first, z_i and z_j will be split into separate lists (and hence will never be compared)

More Questions

- Q: What is

  $P(\text{either } z_i \text{ or } z_j \text{ are first elems in } Z_{i,j} \text{ chosen as pivots})$

- A: $P(z_i \text{ chosen as first elem in } Z_{i,j}) +$ $P(z_j \text{ chosen as first elem in } Z_{i,j})$
- Further note that number of elems in $Z_{i,j}$ is $j - i + 1$, so

$$P(z_i \text{ chosen as first elem in } Z_{i,j}) = \frac{1}{j-i+1}$$

and

$$P(z_j \text{ chosen as first elem in } Z_{i,j}) = \frac{1}{j-i+1}$$

- Hence

$$P(z_i \text{ or } z_j \text{ are first elems in } Z_{i,j} \text{ chosen as pivots}) = \frac{2}{j-i+1}$$

Conclusion

$$\begin{aligned} E(X_{i,j}) &= P(z_i \text{ is compared to } z_j) && (1) \\ &= \frac{2}{j-i+1} && (2) \end{aligned}$$

Putting it together

$$\begin{aligned} E(X) &= E\left(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}\right) && (3) \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{i,j}) && (4) \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} && (5) \\ &= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} && (6) \\ &< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} && (7) \\ &= \sum_{i=1}^{n-1} O(\log n) && (8) \\ &= O(n \log n) && (9) \end{aligned}$$

Questions

- Q: Why is $\sum_{k=1}^{n} \frac{2}{k} = O(\log n)$?
- A:

$$\sum_{k=1}^{n} \frac{2}{k} = 2 \sum_{k=1}^{n} 1/k \quad (10)$$

$$\leq 2(\ln n + 1) \quad (11)$$

- Where the last step follows by an integral bound on the sum (p. 1067)

Take Away

- The expected number of comparisons for r-quicksort is $O(n \log n)$
- Competitive with mergesort and heapsort
- Randomized version is "better" than deterministic version

How Fast Can We Sort?

- Q: What is a lowerbound on the runtime of any sorting algorithm?
- We know that $\Omega(n)$ is a trivial lowerbound
- But all the algorithms we've seen so far are $O(n \log n)$ (or $O(n^2)$), so is $\Omega(n \log n)$ a lowerbound?

Comparison Sorts

- Definition: An sorting algorithm is a *comparison sort* if the sorted order they determine is based only on comparisons between input elements.
- Heapsort, mergesort, quicksort, bubblesort, and insertion sort are all comparison sorts
- We will show that any comparison sort must take $\Omega(n \log n)$

Comparisons

- Assume we have an input sequence $A = (a_1, a_2, \ldots, a_n)$
- In a comparison sort, we only perform tests of the form $a_i < a_j$, $a_i \leq a_j$, $a_i = a_j$, $a_i \geq a_j$, or $a_i > a_j$ to determine the relative order of all elements in A
- We'll assume that all elements are distinct, and so note that the only comparison we need to make is $a_i \leq a_j$.
- This comparison gives us a yes or no answer

Decision Tree Model

- A decision tree is a full binary tree that gives the possible sequences of comparisons made for a particular input array, A
- Each internal node is labelled with the indices of the two elements to be compared
- Each leaf node gives a permutation of A

Decision Tree Model

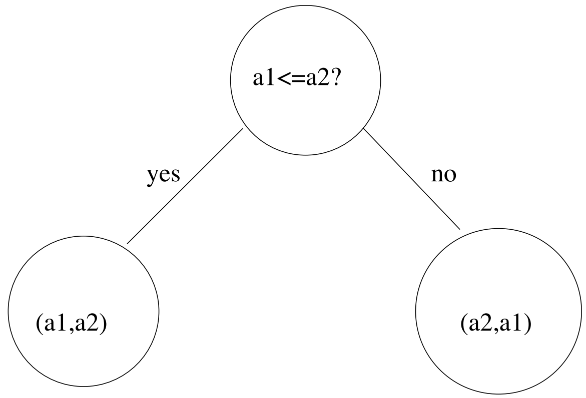
- The execution of the sorting algorithm corresponds to a path from the root node to a leaf node in the tree.
- We take the left child of the node if the comparison is $\leq$ and we take the right child if the comparison is $>$
- The internal nodes along this path give the comparisons made by the alg, and the leaf node gives the output of the sorting algorithm.

Leaf Nodes

- Any correct sorting algorithm must be able to produce each possible permutation of the input
- Thus there must be at least $n!$ leaf nodes
- The length of the longest path from the root node to a leaf in this tree gives the worst case run time of the algorithm (i.e. the height of the tree gives the worst case runtime)

Example

- Consider the problem of sorting an array of size two: $A = (a_1, a_2)$
- Following is a decision tree for this problem.



In-Class Exercise

- Give a decision tree for sorting an array of size three: $A = (a_1, a_2, a_3)$
- What is the height? What is the number of leaf nodes?

Height of Decision Tree

- Q: What is the height of a binary tree with at least $n!$ leaf nodes?
- A: If h is the height, we know that $2^h \geq n!$
- Taking log of both sides, we get $h \geq \log(n!)$

Height of Decision Tree

- Q: What is $\log(n!)$?
- A: It is

$$\begin{aligned}
\log(n*(n-1)*\cdots*1) &= \log n + \log(n-1) + \cdots + \log 1 \\
&\geq (n/2)\log(n/2) \\
&\geq (n/2)(\log n - \log 2) \\
&= \Omega(n \log n)
\end{aligned}$$

- Thus any decision tree for sorting n elements will have a height of $\Omega(n \log n)$

Take Away

- We've just proven that any comparison-based sorting algorithm takes $\Omega(n \log n)$ time
- This does *not* mean that *all* sorting algorithms take $\Omega(n \log n)$ time
- In fact, there are non comparison-based sorting algorithms which, under certain circumstances, are asymptotically faster.

Bucket Sort

- Bucket sort assumes that the input is drawn from a uniform distribution over the range $[0, 1)$
- Basic idea is to divide the interval $[0, 1)$ into n equal size regions, or buckets
- We expect that a small number of elements in A will fall into each bucket
- To get the output, we can sort the numbers in each bucket and just output the sorted buckets in order

Bucket Sort

```
//PRE: A is the array to be sorted, all elements in A[i] are between
$0$ and $1$ inclusive.
//POST: returns a list which is the elements of A in sorted order
BucketSort(A){
B = new List[]
n = length(A)
for (i=1;i<=n;i++){
  insert A[i] at end of list B[floor(n*A[i])];
}
for (i=0;i<=n-1;i++){
  sort list B[i] with insertion sort;
}
return the concatenated list B[0],B[1],...,B[n-1];
}
```


Bucket Sort

- Claim: If the input numbers are distributed uniformly over the range $[0, 1)$, then Bucket sort takes expected time $O(n)$
- Let $T(n)$ be the run time of bucket sort on a list of size n
- Let n_i be the random variable givingthe number of elements in bucket $B[i]$
- Then $T(n) = \Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)$

Analysis

- We know $T(n) = \Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)$
- Taking expectation of both sides, we have

$$\begin{aligned} E(T(n)) &= E(\Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)) \\ &= \Theta(n) + \sum_{i=0}^{n-1} E(O(n_i^2)) \\ &= \Theta(n) + \sum_{i=0}^{n-1} (O(E(n_i^2))) \end{aligned}$$

- The second step follows by linearity of expectation
- The last step holds since for any constant a and random variable X, $E(aX) = aE(X)$ (see Equation C.21 in the text)

Analysis

- We claim that $E(n_i^2) = 2 - 1/n$
- To prove this, we define indicator random variables: $X_{ij} = 1$ if $A[j]$ falls in bucket i and 0 otherwise (defined for all i, $0 \le i \le n-1$ and j, $1 \le j \le n$)
- Thus, $n_i = \sum_{j=1}^{n} X_{ij}$
- We can now compute $E(n_i^2)$ by expanding the square and regrouping terms

Analysis

$$\begin{aligned} E(n_{i^2}) &= E((\sum_{j=1}^{n} X_{ij})^2) \\ &= E(\sum_{j=1}^{n} \sum_{k=1}^{n} X_{ij} X_{ik}) \\ &= E(\sum_{j=1}^{n} X_{ij}^2 + \sum_{1 \le j \le n} \sum_{1 \le k \le n, k \ne j} X_{ij} X_{ik}) \\ &= \sum_{j=1}^{n} E(X_{ij}^2) + \sum_{1 \le j \le n} \sum_{1 \le k \le n, k \ne j} E(X_{ij} X_{ik})) \end{aligned}$$

Analysis

- We can evaluate the two summations separately. X_{ij} is 1 with probability $1/n$ and 0 otherwise
- Thus $E(X_{ij}^2) = 1 * (1/n) + 0 * (1 - 1/n) = 1/n$
- Where $k \ne j$, the random variables X_{ij} and X_{ik} are independent
- For any two *independent* random variables X and Y, $E(XY) = E(X)E(Y)$ (see C.3 in the book for a proof of this)
- Thus we have that

$$\begin{aligned} E(X_{ij} X_{ik}) &= E(X_{ij}) E(X_{ik}) \\ &= (1/n)(1/n) \\ &= (1/n^2) \end{aligned}$$

Analysis

- Substituting these two expected values back into our main equation, we get:

$$\begin{aligned} E(n_i^2) &= \sum_{j=1}^{n} E(X_{ij}^2) + \sum_{1\le j\le n}\ \sum_{1\le k\le n, k\ne j} E(X_{ij}X_{ik})) \\ &= \sum_{j=1}^{n}(1/n) + \sum_{1\le j\le n}\ \sum_{1\le k\le n,k\ne j}(1/n^2) \\ &= n(1/n) + (n)(n-1)(1/n^2) \\ &= 1+(n-1)/n \\ &= 2-(1/n) \end{aligned}$$

Analysis

- Recall that $E(T(n)) = \Theta(n) + \sum_{i=0}^{n-1}(O(E(n_i^2)))$
- We can now plug in the equation $E(n_i^2) = 2-(1/n)$ to get

$$\begin{aligned} E(T(n)) &= \Theta(n) + \sum_{i=0}^{n-1} 2-(1/n) \\ &= \Theta(n)+\Theta(n) \\ &= \Theta(n) \end{aligned}$$

- Thus the entire bucket sort algorithm runs in expected linear time

Dictionary ADT

A dictionary ADT implements the following operations

- *Insert(x)*: puts the item x into the dictionary
- *Delete(x)*: deletes the item x from the dictionary
- *IsIn(x)*: returns true iff the item x is in the dictionary

Dictionary ADT

- Frequently, we think of the items being stored in the dictionary as *keys*
- The keys typically have *records* associated with them which are carried around with the key but not used by the ADT implementation
- Thus we can implement functions like:
  - *Insert(k,r)*: puts the item (k,r) into the dictionary if the key k is not already there, otherwise returns an error
  - *Delete(k)*: deletes the item with key k from the dictionary
  - *Lookup(k)*: returns the item (k,r) if k is in the dictionary, otherwise returns null

Implementing Dictionaries

- The simplest way to implement a dictionary ADT is with a linked list
- Let l be a linked list data structure, assume we have the following operations defined for l
  - head(l): returns a pointer to the head of the list
  - next(p): given a pointer p into the list, returns a pointer to the next element in the list if such exists, null otherwise
  - previous(p): given a pointer p into the list, returns a pointer to the previous element in the list if such exists, null otherwise
  - key(p): given a pointer into the list, returns the key value of that item
  - record(p): given a pointer into the list, returns the record value of that item

At-Home Exercise

Implement a dictionary with a linked list

- Q1: Write the operation Lookup(k) which returns a pointer to the item with key k if it is in the dictionary or null otherwise
- Q2: Write the operation Insert(k,r)
- Q3: Write the operation Delete(k)
- Q4: For a dictionary with n elements, what is the runtime of all of these operations for the linked list data structure?
- Q5: Describe how you would use this dictionary ADT to count the number of occurences of each word in an online book.

Dictionaries

- This linked list implementation of dictionaries is very slow
- Q: Can we do better?
- A: Yes, with hash tables, AVL trees, etc

Hash Tables

Hash Tables implement the Dictionary ADT, namely:

- Insert(x) - $O(1)$ expected time, $\Theta(n)$ worst case
- Lookup(x) - $O(1)$ expected time, $\Theta(n)$ worst case
- Delete(x) - $O(1)$ expected time, $\Theta(n)$ worst case

Direct Addressing

- Suppose universe of keys is $U = \{0, 1, \ldots, m-1\}$, where m is not too large
- *Assume no two elements have the same key*
- We use an array $T[0..m-1]$ to store the keys
- Slot k contains the elem with key k

Direct Address Functions

```
DA-Search(T,k){ return T[k];}
DA-Insert(T,x){ T[key(x)] = x;}
DA-Delete(T,x){ T[key(x)] = NIL;}
```

Each of these operations takes $O(1)$ time

Direct Addressing Problem

- If universe U is large, storing the array T may be impractical
- Also much space can be wasted in T if number of objects stored is small
- Q: Can we do better?
- A: Yes we can trade time for space