University of Illinois at Urbana-Champaign
Department of Computer Science

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We now know enough to solve the recurrence for Fibonacci Numbers. Specifically, we can notice that the recurrence $F_0 = 0, F_1 = 1, F_i = F_{i-1} + F_{i-2}$ is annihilated by $\mathbf{E}^2 - \mathbf{E} - 1$:

$$\begin{aligned}(\mathbf{E}^2 - \mathbf{E} - 1)\langle F_i\rangle &= \mathbf{E}^2\langle F_i\rangle - \mathbf{E}\langle F_i\rangle - \langle F_i\rangle \\ &= \langle F_{i+2}\rangle - \langle F_{i+1}\rangle - \langle F_i\rangle \\ &= \langle F_{i+2} - F_{i+1} - F_i\rangle \\ &= \langle 0\rangle\end{aligned}$$

If we factor $\mathbf{E}^2 - \mathbf{E} - 1$ into its roots (using the Quadratic Equation) we get:

$$\mathbf{E}^2 - \mathbf{E} - 1 = (E - \phi)(E - \hat{\phi})$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$.

Thus, we know that $(E - \phi)(E - \hat{\phi})$ annihilates $\langle F_i\rangle$ and this means that F_i is of the form:

$$F_i = c\phi^i + \hat{c}\hat{\phi}^i$$

In order to get the constants c and $\hat{c}$, we analyze the initial conditions conditions $F_0 = 0, F_1 = 1$, which give us the set of equations

$$\begin{aligned} F_0 : \qquad 0 &= c + \hat{c} \\ F_1 : \qquad 1 &= c\left(\frac{1+\sqrt{5}}{2}\right) + \hat{c}\left(\frac{1-\sqrt{5}}{2}\right)\end{aligned}$$

We have 2 equations and 2 unknowns, and we solve them to get $c = \frac{1}{\sqrt{5}}$ and $\hat{c} = -\frac{1}{\sqrt{5}}$, so that our explicit formula for the i'th Fibonacci number is:

$$F_i = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^i$$

It is quite amazing that F_i turns out to be an integer, with all the square roots in its formula; however, if we do all the math correctly, we will see that all the square roots will always cancel out or disappear when we plug in integral values for i.

0.1 Degenerate Cases

Before we decide that we can now solve all recurrences, we should note that in our above formulation of $(\mathbf{E} - a)(\mathbf{E} - b)$, we assumed that $a \neq b$. What does the sequence $(\mathbf{E} - a)(\mathbf{E} - a) = (\mathbf{E} - a)^2$ annihilate? It turns out that this degenerate sequence will annihilate sequences such as $\langle ia^i\rangle$, as we can see:

$$\begin{aligned}(\mathbf{E} - a)\langle ia^i\rangle &= \langle (i+1)a^{i+1} - (a)ia^i\rangle \\ &= \langle (i+1)a^{i+1} - ia^{i+1}\rangle\end{aligned}$$

$$
\begin{aligned}
& = \langle (i+1-i)a^{i+1} \rangle \\
& = \langle a^{i+1} \rangle \\
(\mathbf{E}-a)^2 \langle ia^i \rangle & = (\mathbf{E}-a)\langle a^{i+1} \rangle \\
& = \langle 0 \rangle
\end{aligned}
$$

From this we can generalize to:

$$(\mathbf{E}-a)^n \text{ annihilates } \langle p(i)a^i \rangle \tag{1}$$

where $p(i)$ is any polynomial in i of degree $n-1$. As an example, $(\mathbf{E}-1)^3$ annihilates the sequence $\langle i^2 \times 1^i \rangle = \langle i^2 \rangle = \langle 1, 4, 9, 16, 25, \ldots \rangle$, since $p(i) = i^2$ is a polynomial of degree $n-1 = 2$.

As a review, try to explain the answers to the following questions:

Q: What does $(\mathbf{E}-3)(\mathbf{E}-2)(\mathbf{E}-1)$ annihilate?

A: $c_1 1^k + c_2 2^k + c_3 3^k$

Q: What does $(\mathbf{E}-3)^2(\mathbf{E}-2)(\mathbf{E}-1)$ annihilate?

A: $c_1 1^k + c_2 2^k + (c_3 k + c_4) 3^k$

0.2 Summary

In summary, we have learned the following things about sequences:

Consider two sequences of numbers:
$A = \langle a_0, a_1, a_2, a_3, a_4, \ldots \rangle$ and $B = \langle b_0, b_1, b_2, b_3, b_4, \ldots \rangle$.
We can multiply by a constant. $cA = \langle ca_0, ca_1, ca_2, ca_3, ca_4, \ldots \rangle$
We can shift to the left: $\mathbf{E}A = \langle a_1, a_2, a_3, a_4, \ldots \rangle$
We can shift n positions to the left: $\mathbf{E}^n A = \langle a_n, a_{n+1}, a_{n+2}, a_{n+3}, \ldots \rangle$
We can add two sequences: $A + B = \langle a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, \ldots \rangle$

Notice that we do not have a multiplication operator for two sequences. Multiplication of two sequences is usually accomplished by *convolution*, which is a more elaborate process than we need for our purposes so far.

We have also learned some things about annihilators.

$(\mathbf{E}-a)$ annihilates only all sequences of the form $\langle c_0 a^i \rangle$
$(\mathbf{E}-a)(\mathbf{E}-b)$ annihilates only all sequences of the form $\langle c_0 a^i + c_1 b^i \rangle$
$(\mathbf{E}-a_0)(\mathbf{E}-a_1)\ldots(\mathbf{E}-a_n)$ annihilates only sequences of the form $\langle c_0 a_0^i + c_1 a_1^i + \ldots c^n a_n^i \rangle$, here $a_i \neq a_j$, when $i \neq j$
$(\mathbf{E}-a)^2$ annihilates only sequences of the form $\langle (c_0 i + c_1) a^i \rangle$
$(\mathbf{E}-a)^n$ annihilates only sequences of the form $\langle p(i) a^i \rangle$, $degree(p(i)) = n-1$

The general expressions in the annihilator box above are really the most important things to remember about annihilators because they help you to solve any recurrence for which you can write down an annihilator. The general method is:

- write down the annihilator for the recurrence
- factor the annihilator
- determine what sequence each factor annihilates
- put the sequences together

- solve for constants of the solution by using initial conditions

For example, consider the recurrence R:

$$\begin{aligned} r_0 &= 1; r_1 = 5; r_2 = 17 \\ r_i &= 7r_{i-1} - 16r_{i-2} + 12r_{i-3} \end{aligned}$$

We go through the steps described above:

- **write down the annihilator.** From the definition of the recurrence, we can see that $\mathbf{E}^3 - 7\mathbf{E}^2 + 16\mathbf{E} - 12$ will annihilate the recurrence since $r_i - 7r_{i-1} + 16r_{i-2} - 12r_{i-3} = 0$.
- **factor.** We can factor the annihilator by hand or our favorite program to get: $\mathbf{E}^3 - 7\mathbf{E}^2 + 16\mathbf{E} - 12 = (\mathbf{E} - 2)^2(\mathbf{E} - 3)$
- **determine sequences.** Our little chart above shows that $(\mathbf{E} - 2)^2$ annihilates $\langle (ai + b) \times 2^i \rangle$ and $(\mathbf{E} - 3)$ annihilates $\langle d \times 3^i \rangle$
- **put together.** Putting together the two sequences gives that $(\mathbf{E} - 2)^2(\mathbf{E} - 3)$ annihilates $\langle (ai + b) \times 2^i + d \times 3^i \rangle$
- **solve.** Since we know that $r_0 = 1; r_1 = 5; r_2 = 17$, we have three equations in three unknowns:

$$\begin{aligned} 1 &= (a * 0 + b)2^0 + d3^0 = b + d \\ 5 &= (a * 1 + b)2^1 + d3^1 = 2a + 2b + 3d \\ 17 &= (a * 2 + b)2^2 + d3^2 = 8a + 4b + 9d \end{aligned}$$

  We can solve these equations to get $a = 1$, $b = 0$, $d = 1$.

Our final solution to the recurrence is thus $r_i = i2^i + 3^i$, which we can verify by hand.

0.3 Solving recurrences containing non-homogeneous parts

We continue with the following observation:

$$(\mathbf{E} - a)^k \text{ annihilates } \begin{pmatrix} \text{polynomial in } n \\ \text{of degree } k - 1 \end{pmatrix} a^n$$

This turns out to be very important. Consider the example of height-balanced trees. Let a_n be the smallest number of nodes needed to obtain a height balanced tree of height n.

One of the subtrees must have a height of $n - 1$. If it didn't, the tree couldn't have a height of n. The other subtree may have any height, but this is a height-balanced tree. Height-balanced trees have the property that the heights of subtrees differ by at most 1. Therefore the other subtree may have a height of either $n - 1$ or $n - 2$. Since we are counting the smallest number of nodes, it must have height $n - 2$.

This leads to the recurrence equation:

$$a_n = \underbrace{a_{n-1}}_{\text{height } n-1} + \underbrace{a_{n-2}}_{\text{height } n-2} + \underbrace{1}_{\text{root}}$$

We refer to the terms in the equation involving a_i as the **homogeneous** terms and the rest as the **non-homogeneous** terms. We know that $\mathbf{E}^2 - \mathbf{E} - 1$ annihilates the homogeneous part, $a_n = a_{n-1} + a_{n-2}$.

Let us try applying this annihilator to the entire equation:

$$\begin{aligned}(\mathbf{E}^2 - \mathbf{E} - 1)\langle a_i \rangle &= \mathbf{E}^2\langle a_i \rangle - \mathbf{E}\langle a_i \rangle - 1\langle a_i \rangle \\ &= \langle a_{i+2} \rangle - \langle a_{i+1} \rangle - \langle a_i \rangle \\ &= \langle a_{i+2} - a_{i+1} - a_i \rangle \\ &= \langle 1, 1, 1, \ldots \rangle\end{aligned}$$

This is close to what we want, but $\langle 1, 1, 1, \ldots \rangle$ is not quite $\langle 0, 0, 0, \ldots \rangle$. It is clear, though, that $\mathbf{E} - 1$ annihilates $\langle 1, 1, 1, \ldots \rangle$. Thus $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)$ annihilates $a_n = a_{n-1} + a_{n-2} + 1$.

We can easily factor $\mathbf{E}^2 - \mathbf{E} - 1$, and we know what $(\mathbf{E} - c)$ annihilates. Thus $a_n = c_1\phi^n + c_2\hat{\phi}^n + c_3 1^n$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$.

All that remains is to find the constants c, c_2, and c_3. The boundary conditions give us two equations, but we need a third to find three unknowns. We can calculate a_2 from the recurrence to accomplish this.

$$\begin{aligned}a_0 : 1 &= c_1\phi^0 + c_2\hat{\phi}^0 + c_3 \\ &= c_1 + c_2 + c_3 \\ a_1 : 2 &= c_1\phi^1 + c_2\hat{\phi}^1 + c_3 \\ &= c_1\phi + c_2\hat{\phi} + c_3 \\ a_2 : 4 &= c_1\phi^2 + c_2\hat{\phi}^2 + c_3\end{aligned}$$

We can solve for c, c_2, and c_3 from these three equations.

0.4 Some more examples

Consider $a_n = a_{n-1} + a_{n-2} + 2$. The residue is $\langle 2, 2, 2, \ldots \rangle$ and the annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)$. The value of a_2 changes. The equations are now:

$$\begin{aligned}a_0 : 1 &= c_1 + c_2 + c_3 \\ a_1 : 2 &= c\phi + c_2\hat{\phi} + c_3 \\ a_2 : 5 &= c\phi^2 + c_2\hat{\phi}^2 + c_3\end{aligned}$$

Consider $a_n = a_{n-1} + a_{n-2} + 3$. The residue is $\langle 3, 3, 3, \ldots \rangle$ and the annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)$. The value of a_2 changes. The equations are now:

$$\begin{aligned}a_0 : 1 &= c_1 + c_2 + c_3 \\ a_1 : 2 &= c\phi + c_2\hat{\phi} + c_3 \\ a_2 : 6 &= c\phi^2 + c_2\hat{\phi}^2 + c_3\end{aligned}$$

Consider $a_n = a_{n-1} + a_{n-2} + 2^n$. The residue is $\langle 1, 2, 4, 8, \ldots \rangle$ and the annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 2)$. The equations are now:

$$\begin{aligned}a_0 : 1 &= c_1 + c_2 + c_3 \\ a_1 : 2 &= c\phi + c_2\hat{\phi} + 2c_3 \\ a_2 : 4 &= c\phi^2 + c_2\hat{\phi}^2 + 4c_3\end{aligned}$$

Consider $a_n = a_{n-1} + a_{n-2} + n$. The residue is $\langle 1, 2, 3, 4, \ldots \rangle$ and the annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)^2$. The equations are now:

$$\begin{aligned} a_0 : 1 &= c_1 + c_2 + c_4 \\ a_1 : 2 &= c\phi + c_2\hat{\phi} + (c_3 + c_4) \\ a_2 : 4 &= c\phi^2 + c_2\hat{\phi}^2 + (2c_3 + c_4) \\ a_3 : 9 &= c\phi^3 + c_2\hat{\phi}^3 + (3c_3 + c_4) \end{aligned}$$

Consider $a_n = a_{n-1} + a_{n-2} + n^2$. The residue is $\langle 1, 4, 9, 25, \ldots \rangle$ and the annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)^3$. The equations are now:

$$\begin{aligned} a_0 : 1 &= c_1 + c_2 + c_5 \\ a_1 : 2 &= c\phi + c_2\hat{\phi} + (c_3 + c_4 + c_5) \\ a_2 : 4 &= c\phi^2 + c_2\hat{\phi}^2 + (2^2c_3 + 2c_4 + c_5) \\ a_3 : 15 &= c\phi^3 + c_2\hat{\phi}^3 + (3^2c_3 + 3c_4 + c_5) \\ a_4 : 25 &= c\phi^4 + c_2\hat{\phi}^4 + (4^2c_3 + 4c_4 + c_5) \end{aligned}$$

Consider $a_n = a_{n-1} + a_{n-2} + n^2 - 2^n$. The residue is $\langle 1-1, 4-4, 9-8, 25-16, \ldots \rangle$ and the annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)^3(\mathbf{E} - 2)$. The equations can be calculated from the annihilator.

Consider $a_n = a_{n-1} + a_{n-2} + \phi^n$. The annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - \phi) = (\mathbf{E} - \phi)^2(\mathbf{E} - \hat{\phi})$. The equations can be calculated from the annihilator. (Other recurrence solving methods will have a "inteference" problem with this equation, while the operator method does not.)

Our method does not work on $a_n = a_{n-1} + \frac{1}{n}$ or $a_n = a_{n-1} + \lg n$, as $\frac{1}{n}$ and $\lg n$ do not have annihilators. Our tool, as it stands, is limited