CS 561, Lecture 9

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Today's Outline

- Minimum Spanning Trees
- Safe Edge Theorem
- Kruskal and Prim's algorithms
- Graph Representation

Graph Definition

- A graph is a pair of sets (V, E).
- We call V the vertices of the graph
- E is a set of vertex pairs which we call the edges of the graph.
- In an *undirected* graph, the edges are unordered pairs of vertices and in a *directed* graph, the edges are ordered pairs.
- We assume that there is never an edge from a vertex to itself (no self-loops) and that there is at most one edge from any vertex to any other (no multi-edges)
- $|V|$ is the number of vertices in the graph and $|E|$ is the number of edges

Graph Defns

- A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$
- If (u, v) is an edge in a graph, then u is a *neighbor* of v
- For a vertex v, the *degree* of v, $deg(v)$, is equal to the number of neighbors of v
- A *path* is a sequence of edges, where each successive pair of edges shares a vertex
- A graph is *connected* if there is a path from any vertex to any other vertex
- A disconnected graph consists of several *connected components* which are maximal connected subgraphs
- Two vertices are in the same component if and only if there is a path between them

Graph Defns

For undirected graphs:

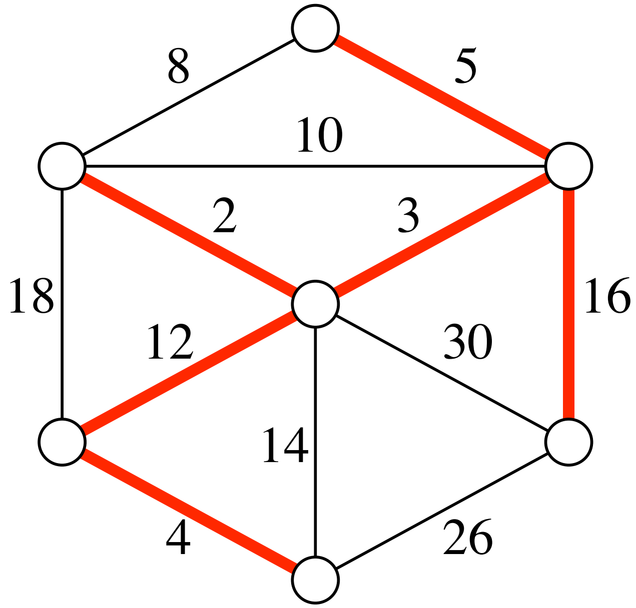
- A *cycle* is a path that starts and ends at the same vertex and has at least 3 edges
- A graph is *acyclic* if no subgraph is a cycle. Acyclic graphs are also called *forests*
- A *tree* is a connected acyclic graph. It's also a connected component of a forest.
- A *spanning tree* of a graph G is a subgraph that is a tree and also contains every vertex of G. A graph can only have a spanning tree if it's connected
- A *spanning forest* of G is a collection of spanning trees, one for each connected component of G

Minimum Spanning Tree Problem

- Suppose we are given a connected, undirected *weighted* graph
- That is a graph $G = (V, E)$ together with a function $w\colon E \to R$ that assigns a *weight* $w(e)$ to each edge e. (We assume the weights are real numbers)
- Our task is to find the *minimum spanning tree* of G, i.e., the spanning tree T minimizing the function

$$w(T) = \sum_{e \in T} w(e)$$

Example



A weighted graph and its minimum spanning tree

Applications

- Creating an inexpensive road network to connect cities
- Wiring up homes for phone service with the smallest amount of wire
- Finding a good approximation to the TSP problem

Generic MST Algorithm

```
Generic-MST(G,w){
  A = {};
  while (A does not form a spanning tree){
    find an edge (u,v) that is safe for A;
    A = A union (u,v);
  }
return A;
}
```

Safe edges

- A *cut* $(S, V - S)$ of a graph $G = (V, E)$ is a partition of V
- An edge (u, v) *crosses* the cut $(S, V - S)$ if one of its endpoints is in S and the other is in $V - S$
- A cut *respects* a set of edges A if no edge in A crosses the cut.
- An edge is a *light edge* crossing a cut if its weight is the minimum of any edge crossing the cut

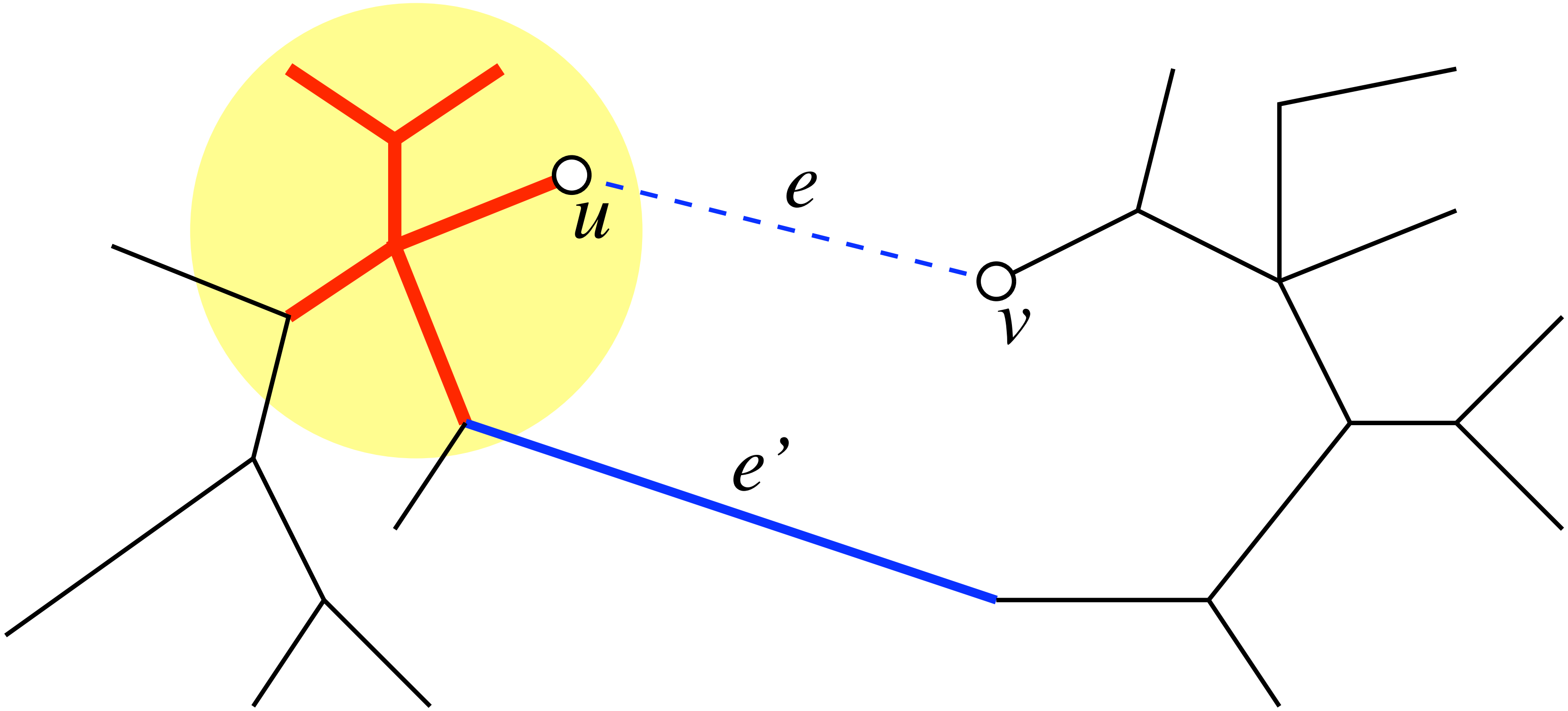
Theorem

Let $G = (V, E)$ *be a connected, undirected graph with a real-valued weight function* w *defined on* E*. Let* A *be a subset of* E *that is included in some minimum spanning tree for* G*. Let* $(S, V - S)$ *be any cut of* G *that respects* A *and let* (u, v) *be a light edge crossing* $(S, V - S)$*. Then edge* (u, v) *is safe for* A

Proof

- Let T be a MST that includes some set of edges A
- Assume that T does not contain the light edge $e = (u, v)$
- Since T is connected, it contains a unique path from u to v and at least one edge e' on this path crosses the cut that respects A
- Note that $w(e) \leq w(e')$ by assumption
- Removing e' from the MST and adding e gives us a new spanning tree T'
- T' has total weight no more than T and this T' must also be a MST. QED.

Example



Proof that every safe edge is in some MST. The red edges are the set A.

Corollary

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, and let $C = (V_c, E_c)$ be a connected component (tree) in the forest $G_A = (V, A)$. If (u, v) is a light edge connecting C to some other component in G_A, then (u, v) is safe for A

Proof: The cut $(V_C, V - V_C)$ respects A, and (u, v) is a light edge for this cut. Therefore (u, v) is safe for A.

Two MST algorithms

- There are two major MST algorithms, Kruskal's and Prim's
- In Kruskal's algorithm, the set A is a forest. The safe edge added to A is always a least-weighted edge in the graph that connects two distinct components
- In Prim's algorithm, the set A forms a single tree. The safe edge added to A is always a least-weighted edge connecting the tree to a vertex not in the tree

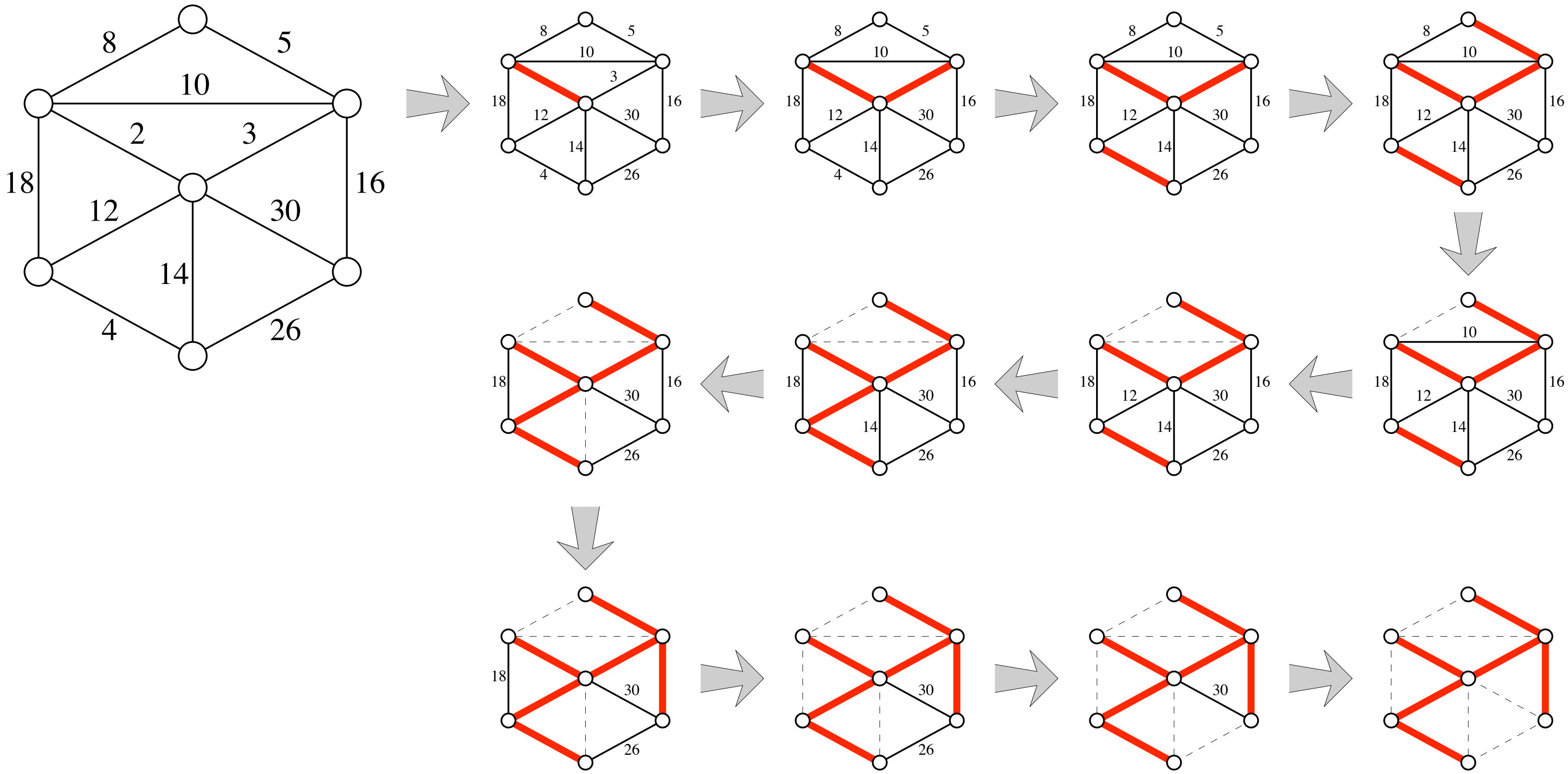
Kruskal's Algorithm

- Q: In Kruskal's algorithm, how do we determine whether or not an edge connects two distinct connected components?
- A: We need some way to keep track of the sets of vertices that are in each connected components and a way to take the union of these sets when adding a new edge to A merges two connected components
- What we need is the data structure for maintaining disjoint sets (aka Union-Find) that we discussed last week

Kruskal's Algorithm

```
MST-Kruskal(G,w){
  for (each vertex v in V)
    Make-Set(v);
  sort the edges of E into nondecreasing order by weight;
  for (each edge (u,v) in E taken in nondecreasing order){
    if(Find-Set(u)!=Find-Set(v)){
      A = A union (u,v);
      Set-Union(u,v);
    }
  }
  return A;
}
```

Example Run



Kruskal's algorithm run on the example graph. Thick edges are in A. Dashed edges are useless.

Correctness?

- Correctness of Kruskal's algorithm follows immediately from the corollary
- Each time we add the lightest weight edge that connects two connected components, hence this edge must be safe for A
- This implies that at the end of the algorith, A will be a MST

Runtime?

- The runtime for Kruskal's alg. will depend on the implementation of the disjoint-set data structure. We'll assume the implementation with union-by-rank and path-compression which we showed has amortized cost of $\log^* n$

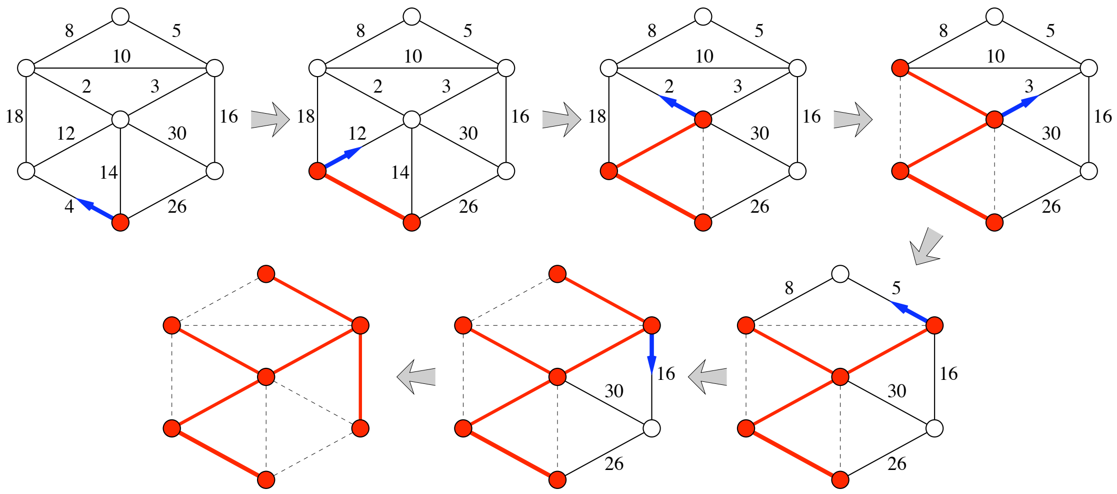
Runtime?

- Time to sort the edges is $O(|E| \log |E|)$
- Total amount of time for the $|V|$ Make-Sets and up to $|E|$ Set-Unions is $O((|V| + |E|) \log^* |V|)$
- Since G is connected, $|E| \geq |V| - 1$ and so $O((|V| + |E|) \log^* |V|) = O(|E| \log^* |V|) = O(|E| \log |E|)$
- Total amount of additional work done in the for loop is just $O(E)$
- Thus total runtime of the algorithm is $O(|E| \log |E|)$
- Since $|E| \leq |V|^2$, we can rewrite this as $O(|E| \log |V|)$

Prim's Algorithm

- In Prim's algorithm, the set A maintained by the algorithm forms a single tree.
- The tree starts from an arbitrary root vertex and grows until it spans all the vertices in V
- At each step, a light edge is added to the tree A which connects A to an isolated vertex of $G_A = (V, A)$
- By our Corollary, this rule adds only safe edges to A, so when the algorithm terminates, it will return a MST

Example Run



Prim's algorithm run on the example graph, starting with the bottom vertex.

At each stage, thick edges are in A, an arrow points along A's safe edge, and dashed edges are useless.

An Implementation

- To implement Prim's algorithm, we keep all edges adjacent to A in a heap
- When we pull the minimum-weight edge off the heap, we first check to see if both its endpoints are in A
- If not, we add the edge to A and then add the neighboring edges to the heap
- If we implement Prim's algorithm this way, its running time is $O(|E| \log |E|) = O(|E| \log |V|)$
- However, we can do better

Prim's Algorithm

- We can speed things up by noticing that the algorithm visits each vertex only once
- Rather than keeping the edges in the heap, we will keep a heap of vertices, where the key of each vertex v is the weight of the minimum-weight edge between v and A (or infinity if there is no such edge)
- Each time we add a new edge to A, we may need to decrease the key of some neighboring vertices

Prim's

We will break up the algorithm into two parts, Prim-Init and Prim-Loop

```
Prim(V,E,s){
  Prim-Init(V,E,s);
  Prim-Loop(V,E,s);
}
```

Prim-Init

```
Prim-Init(V,E,s){
  for each vertex v in V - {s}{
    if ((v,s) is in E){
      edge(v) = (v,s);
      key(v) = w((v,s));
    }else{
      edge(v) = NULL;
      key(v) = infinity;
    }
  Heap-Insert(v);
  }
  Heap-Insert(s);
}
```

Prim-Loop

```
Prim-Loop(V,E,s){
  A = {};
  for (i = 1 to |V| - 1){
    v = Heap-ExtractMin();
    add edge(v) to A;
    for (each edge (u,v) in E){
      if ((u,v) is not in A AND key(u) > w(u,v)){
        edge(u) = (u,v);
        Heap-DecreaseKey(u,w(u,v));
      }
    }
  }
  return A;
}
```

Runtime?

- The runtime of Prim's is dominated by the cost of the heap operations Insert, ExtractMin and DecreaseKey
- Insert and ExtractMin are each called $O(|V|)$ times
- DecreaseKey is called $O(|E|)$ times, at most twice for each edge
- If we use a *Fibonacci Heap*, the amortized costs of Insert and DecreaseKey is $O(1)$ and the amortized cost of ExtractMin is $O(\log |V|)$
- Thus the overall run time of Prim's is $O(|E| + |V| \log |V|)$
- This is faster than Kruskal's unless $E = O(|V|)$

Note

- This analysis assumes that it is fast to find all the edges that are incident to a given vertex
- We have not yet discussed how we can do this
- This brings us to a discussion of how to represent a graph in a computer

Graph Representation

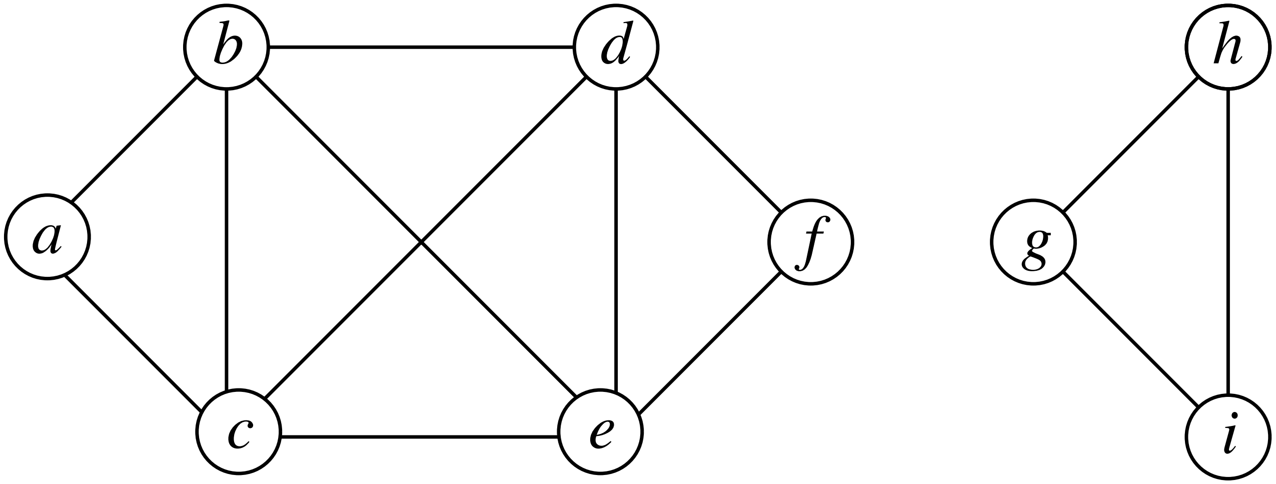
There are two common data structures used to explicity represent graphs

- Adjacency Matrices
- Adjacency Lists

Adjacency Matrix

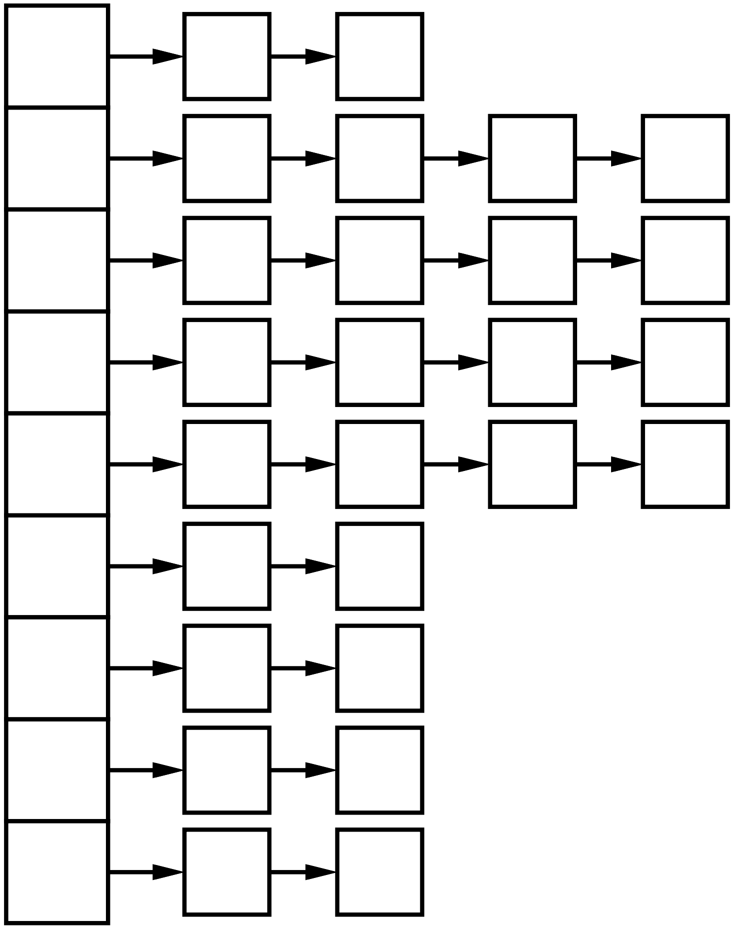
- The adjacency matrix of a graph G is a $|V| \times |V|$ matrix of 0's and 1's
- For an adjacency matrix A, the entry $A[i,j]$ is 1 if $(i,j) \in E$ and 0 otherwise
- For undirectd graphs, the adjacency matrix is always *symmetric*: $A[i,j] = A[j,i]$. Also the diagonal elements $A[i,i]$ are all zeros

Example Graph



Example Representations

$$\begin{array}{c|ccccccccc}
 & a & b & c & d & e & f & g & h & i \\
\hline
a & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
c & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
d & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
e & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
f & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
h & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{array}$$



Adjacency matrix and adjacency list representations for the example graph.

Adjacency Matrix

- Given an adjacency matrix, we can decide in $\Theta(1)$ time whether two vertices are connected by an edge.
- We can also list all the neighbors of a vertex in $\Theta(|V|)$ time by scanning the row corresponding to that vertex
- This is optimal in the worst case, however if a vertex has few neighbors, we still need to examine every entry in the row to find them all
- Also, adjacency matrices require $\Theta(|V|^2)$ space, regardless of how many edges the graph has, so it is only space efficient for very *dense* graphs

Adjacency Lists

- For *sparse* graphs — graphs with relatively few edges — we're better off with adjacency lists
- An adjacency list is an array of linked lists, one list per vertex
- Each linked list stores the neighbors of the corresponding vertex

Adjacency Lists

- The total space required for an adjacency list is $O(|V| + |E|)$
- Listing all the neighbors of a node v takes $O(1+deg(v))$ time
- We can determine if (u, v) is an edge in $O(1 + deg(u))$ time by scanning the neighbor list of u
- Note that we can speed things up by storing the neighbors of a node not in lists but rather in hash tables
- Then we can determine if an edge is in the graph in expected $O(1)$ time and still list all the neighbors of a node v in $O(1+deg(v))$ time