

CS 561, Lecture 9

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Today's Outline

- Minimum Spanning Trees
- Safe Edge Theorem
- Kruskal and Prim's algorithms
- Graph Representation

Graph Definition

- A graph is a pair of sets (V, E) .
- We call V the vertices of the graph
- E is a set of vertex pairs which we call the edges of the graph.
- In an *undirected* graph, the edges are unordered pairs of vertices and in a *directed* graph, the edges are ordered pairs.
- We assume that there is never an edge from a vertex to itself (no self-loops) and that there is at most one edge from any vertex to any other (no multi-edges)
- $|V|$ is the number of vertices in the graph and $|E|$ is the number of edges

Graph Defns

- A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$
- If (u, v) is an edge in a graph, then u is a *neighbor* of v
- For a vertex v , the *degree* of v , $deg(v)$, is equal to the number of neighbors of v
- A *path* is a sequence of edges, where each successive pair of edges shares a vertex
- A graph is *connected* if there is a path from any vertex to any other vertex
- A disconnected graph consists of several *connected components* which are maximal connected subgraphs
- Two vertices are in the same component if and only if there is a path between them

Graph Defns

For undirected graphs:

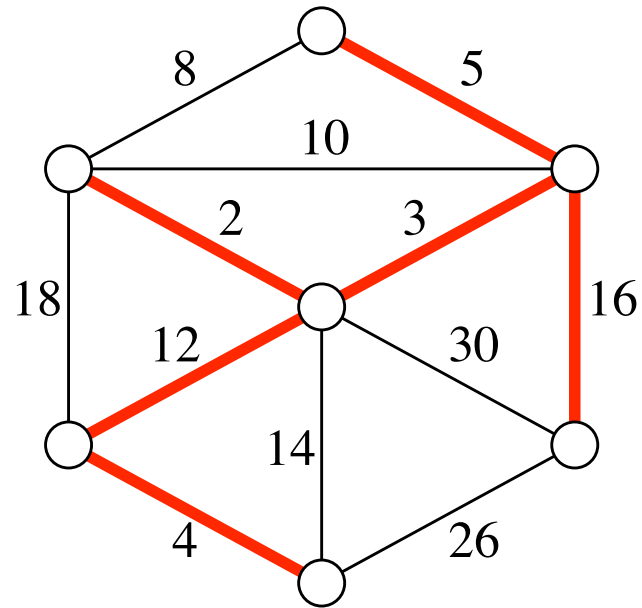
- A *cycle* is a path that starts and ends at the same vertex and has at least 3 edges
- A graph is *acyclic* if no subgraph is a cycle. Acyclic graphs are also called *forests*
- A *tree* is a connected acyclic graph. It's also a connected component of a forest.
- A *spanning tree* of a graph G is a subgraph that is a tree and also contains every vertex of G . A graph can only have a spanning tree if it's connected
- A *spanning forest* of G is a collection of spanning trees, one for each connected component of G

Minimum Spanning Tree Problem

- Suppose we are given a connected, undirected *weighted* graph
- That is a graph $G = (V, E)$ together with a function $w: E \rightarrow \mathbb{R}$ that assigns a *weight* $w(e)$ to each edge e . (We assume the weights are real numbers)
- Our task is to find the *minimum spanning tree* of G , i.e., the spanning tree T minimizing the function

$$w(T) = \sum_{e \in T} w(e)$$

Example



A weighted graph and its minimum spanning tree

Applications

- Creating an inexpensive road network to connect cities
- Wiring up homes for phone service with the smallest amount of wire
- Finding a good approximation to the TSP problem

Generic MST Algorithm

```
Generic-MST(G,w){  
  A = {};  
  while (A does not form a spanning tree){  
    find an edge (u,v) that is safe for A;  
    A = A union (u,v);  
  }  
  return A;  
}
```

Safe edges - Definition

- Let A be any subset of edges in G that is a subset of some MST of G
- An edge e is **safe for** A if $A \cup \{e\}$ is also a subset of a MST.

Safe edges

- A *cut* $(S, V - S)$ of a graph $G = (V, E)$ is a partition of V
- An edge (u, v) *crosses* the cut $(S, V - S)$ if one of its endpoints is in S and the other is in $V - S$
- A cut *respects* a set of edges A if no edge in A crosses the cut.
- An edge is a *light edge* crossing a cut if its weight is the minimum of any edge crossing the cut

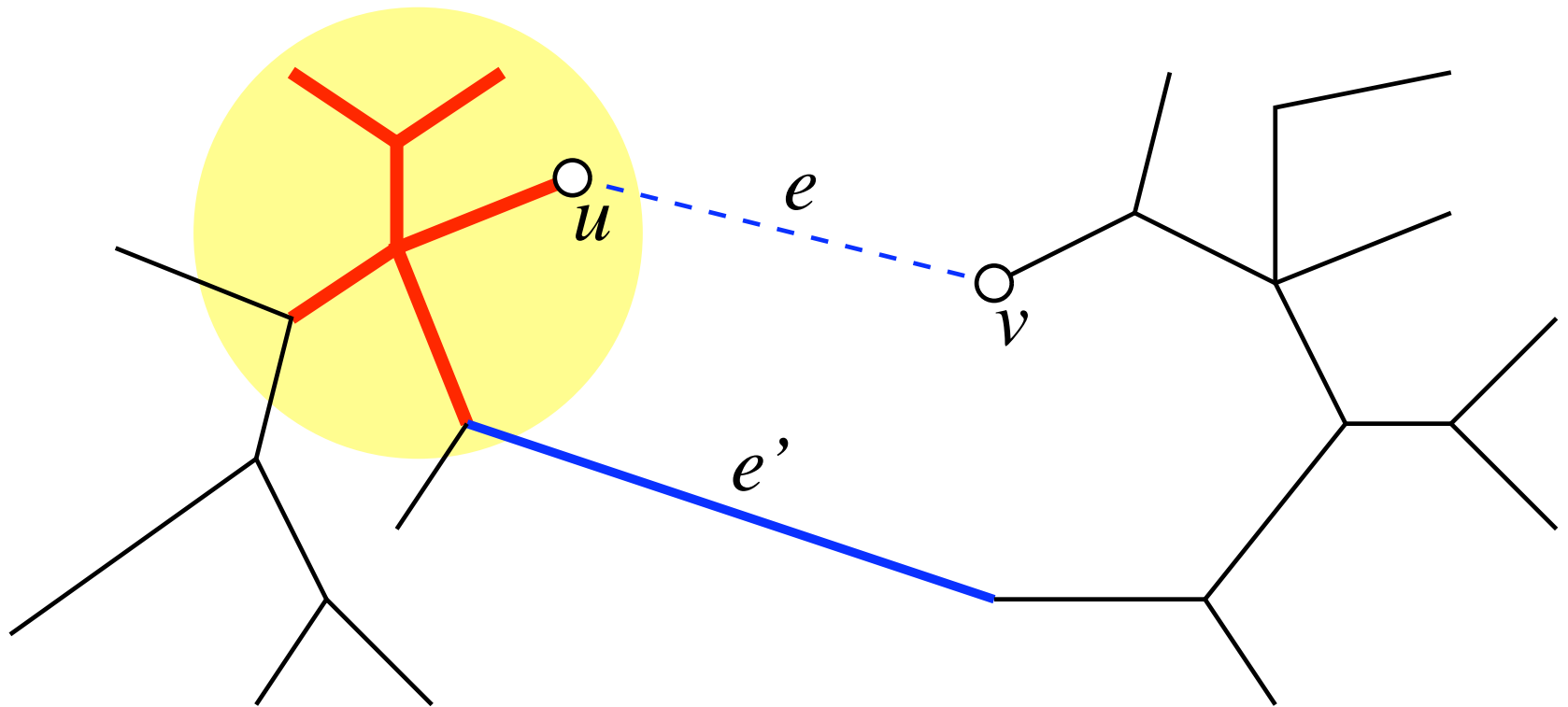
Theorem

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E . Let A be a subset of E that is included in some minimum spanning tree for G . Let $(S, V - S)$ be any cut of G that respects A and let (u, v) be a light edge crossing $(S, V - S)$. Then edge (u, v) is safe for A .

Proof

- Let T be a MST that includes some set of edges A
- Assume that T does not contain the light edge $e = (u, v)$
- Since T is connected, it contains a unique path from u to v and at least one edge e' on this path crosses the cut that respects A
- Note that $w(e) \leq w(e')$ by assumption
- Removing e' from the MST and adding e gives us a new spanning tree T'
- T' has total weight no more than T and this T' must also be a MST. QED.

Example



Proof that every safe edge is in some MST. The red edges are the set A .

Corollary

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E . Let A be a subset of E that is included in some minimum spanning tree for G , and let $C = (V_C, E_C)$ be a connected component (tree) in the forest $G_A = (V, A)$. If (u, v) is a light edge connecting C to some other component in G_A , then (u, v) is safe for A

Proof: The cut $(V_C, V - V_C)$ respects A , and (u, v) is a light edge for this cut. Therefore (u, v) is safe for A .

Two MST algorithms

- There are two major MST algorithms, Kruskal's and Prim's
- In Kruskal's algorithm, the set A is a forest. The safe edge added to A is always a least-weighted edge in the graph that connects two distinct components
- In Prim's algorithm, the set A forms a single tree. The safe edge added to A is always a least-weighted edge connecting the tree to a vertex not in the tree

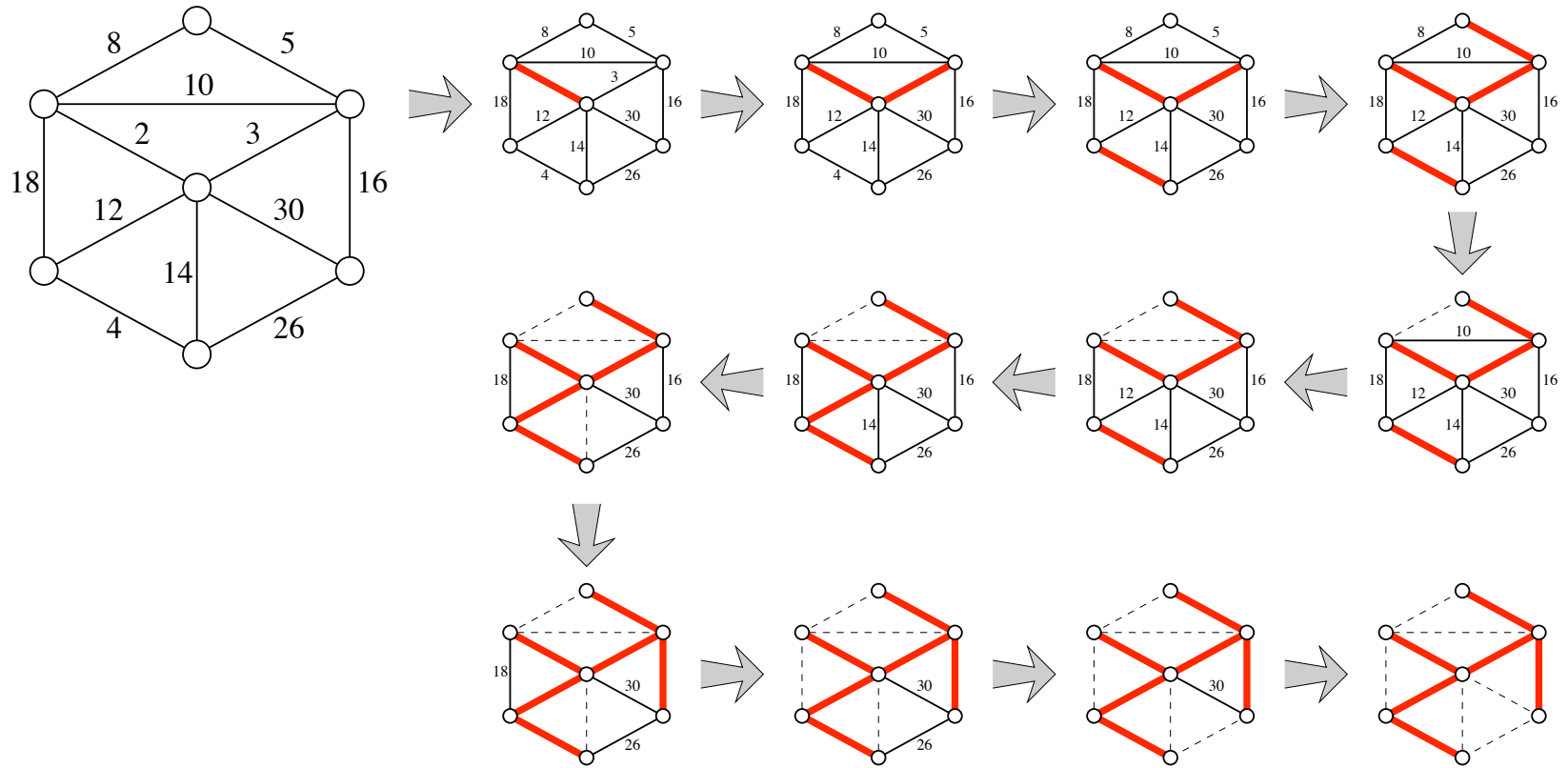
Kruskal's Algorithm

- Q: In Kruskal's algorithm, how do we determine whether or not an edge connects two distinct connected components?
- A: We need some way to keep track of the sets of vertices that are in each connected components and a way to take the union of these sets when adding a new edge to A merges two connected components
- What we need is the data structure for maintaining disjoint sets (aka Union-Find) that we discussed last week

Kruskal's Algorithm

```
MST-Kruskal(G,w){
  for (each vertex v in V)
    Make-Set(v);
  sort the edges of E into nondecreasing order by weight;
  for (each edge (u,v) in E taken in nondecreasing order){
    if(Find-Set(u)!=Find-Set(v)){
      A = A union (u,v);
      Set-Union(u,v);
    }
  }
  return A;
}
```

Example Run



Kruskal's algorithm run on the example graph. Thick edges are in A . Dashed edges are useless.

Correctness?

- Correctness of Kruskal's algorithm follows immediately from the corollary
- Each time we add the lightest weight edge that connects two connected components, hence this edge must be safe for A
- This implies that at the end of the algorithm, A will be a MST

Runtime?

- The runtime for Kruskal's alg. will depend on the implementation of the disjoint-set data structure. We'll assume the implementation with union-by-rank and path-compression which we showed has amortized cost of $\log^* n$

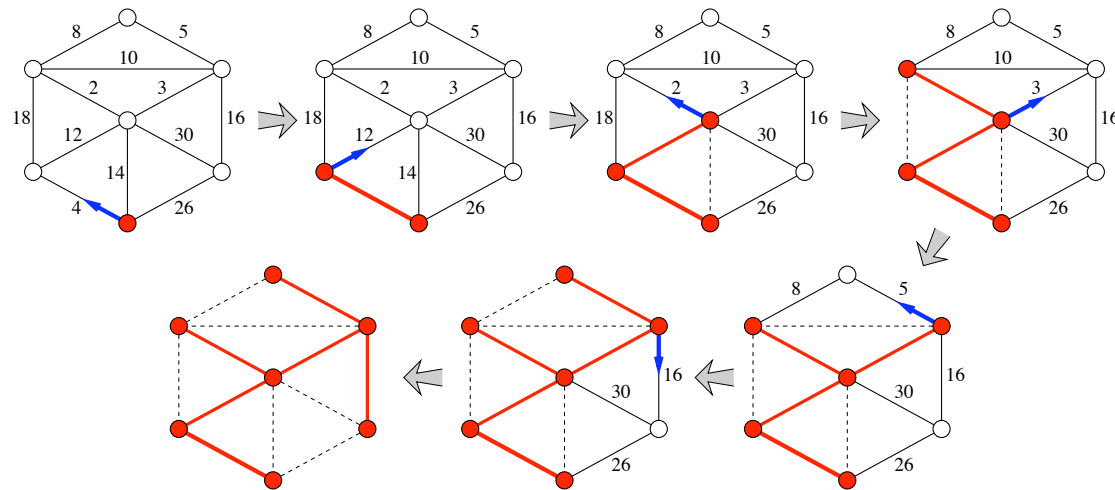
Runtime?

- Time to sort the edges is $O(|E| \log |E|)$
- Total amount of time for the $|V|$ Make-Sets and up to $|E|$ Set-Unions is $O((|V| + |E|) \log^* |V|)$
- Since G is connected, $|E| \geq |V| - 1$ and so $O((|V| + |E|) \log^* |V|) = O(|E| \log^* |V|) = O(|E| \log |E|)$
- Total amount of additional work done in the for loop is just $O(E)$
- Thus total runtime of the algorithm is $O(|E| \log |E|)$
- Since $|E| \leq |V|^2$, we can rewrite this as $O(|E| \log |V|)$

Prim's Algorithm

- In Prim's algorithm, the set A maintained by the algorithm forms a single tree.
- The tree starts from an arbitrary root vertex and grows until it spans all the vertices in V
- At each step, a light edge is added to the tree A which connects A to an isolated vertex of $G_A = (V, A)$
- By our Corollary, this rule adds only safe edges to A , so when the algorithm terminates, it will return a MST

Example Run



Prim's algorithm run on the example graph, starting with the bottom vertex.

At each stage, thick edges are in A , an arrow points along A 's safe edge, and dashed edges are useless.

An Implementation

- To implement Prim's algorithm, we keep all edges adjacent to A in a heap
- When we pull the minimum-weight edge off the heap, we first check to see if both its endpoints are in A
- If not, we add the edge to A and then add the neighboring edges to the heap
- If we implement Prim's algorithm this way, its running time is $O(|E| \log |E|) = O(|E| \log |V|)$
- However, we can do better

Prim's Algorithm

- We can speed things up by noticing that the algorithm visits each vertex only once
- Rather than keeping the edges in the heap, we will keep a heap of vertices, where the key of each vertex v is the weight of the minimum-weight edge between v and A (or infinity if there is no such edge)
- Each time we add a new edge to A , we may need to decrease the key of some neighboring vertices

Prim's

We will break up the algorithm into two parts, Prim-Init and Prim-Loop

```
Prim(V,E,s){  
    Prim-Init(V,E,s);  
    Prim-Loop(V,E,s);  
}
```

Prim-Init

```
Prim-Init(V,E,s){
  for each vertex v in V - {s}{
    if ((v,s) is in E){
      edge(v) = (v,s);
      key(v) = w((v,s));
    }else{
      edge(v) = NULL;
      key(v) = infinity;
    }
  }
  Heap-Insert(v);
}
Heap-Insert(s);
}
```

Prim-Loop

```
Prim-Loop(V,E,s){
  A = {};
  for (i = 1 to |V| - 1){
    v = Heap-ExtractMin();
    add edge(v) to A;
    for (each edge (u,v) in E){
      if ((u,v) is not in A AND key(u) > w(u,v)){
        edge(u) = (u,v);
        Heap-DecreaseKey(u,w(u,v));
      }
    }
  }
  return A;
}
```

Runtime?

- The runtime of Prim's is dominated by the cost of the heap operations Insert, ExtractMin and DecreaseKey
- Insert and ExtractMin are each called $O(|V|)$ times
- DecreaseKey is called $O(|E|)$ times, at most twice for each edge
- If we use a *Fibonacci Heap*, the amortized costs of Insert and DecreaseKey is $O(1)$ and the amortized cost of ExtractMin is $O(\log |V|)$
- Thus the overall run time of Prim's is $O(|E| + |V| \log |V|)$
- This is faster than Kruskal's unless $E = O(|V|)$

Note

- This analysis assumes that it is fast to find all the edges that are incident to a given vertex
- We have not yet discussed how we can do this
- This brings us to a discussion of how to represent a graph in a computer

Graph Representation

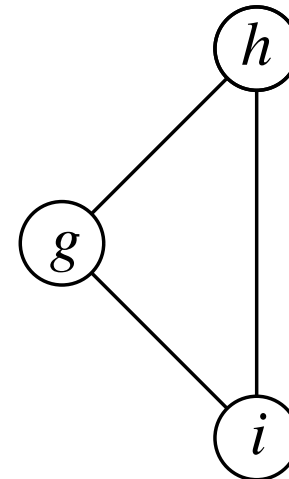
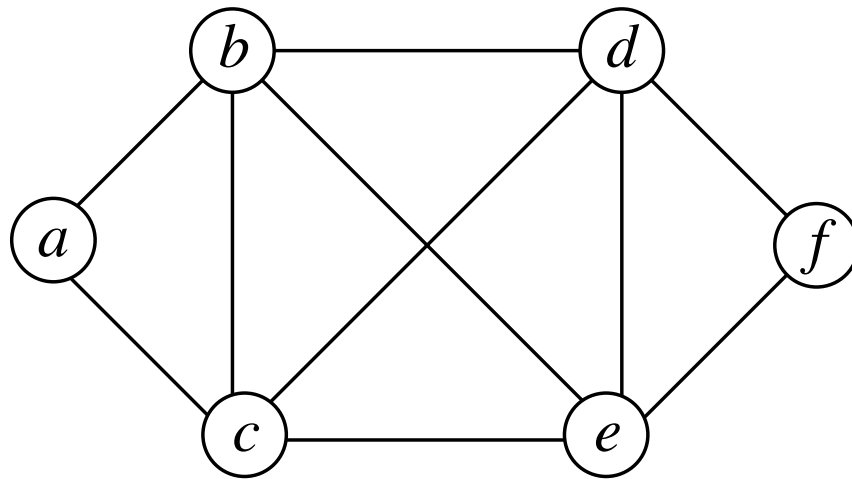
There are two common data structures used to explicitly represent graphs

- Adjacency Matrices
- Adjacency Lists

Adjacency Matrix

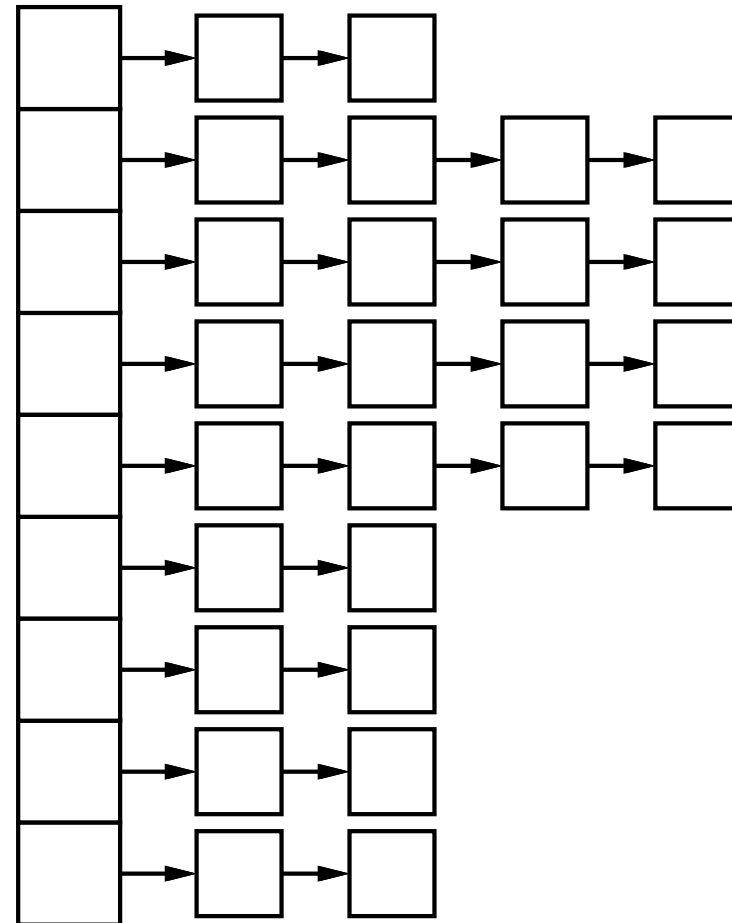
- The adjacency matrix of a graph G is a $|V| \times |V|$ matrix of 0's and 1's
- For an adjacency matrix A , the entry $A[i, j]$ is 1 if $(i, j) \in E$ and 0 otherwise
- For undirected graphs, the adjacency matrix is always *symmetric*: $A[i, j] = A[j, i]$. Also the diagonal elements $A[i, i]$ are all zeros

Example Graph



Example Representations

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>
<i>a</i>	0	1	1	0	0	0	0	0	0
<i>b</i>	1	0	1	1	1	0	0	0	0
<i>c</i>	1	1	0	1	1	0	0	0	0
<i>d</i>	0	1	1	0	1	1	0	0	0
<i>e</i>	0	1	1	1	0	1	0	0	0
<i>f</i>	0	0	0	1	1	0	0	0	0
<i>g</i>	0	0	0	0	0	0	0	1	0
<i>h</i>	0	0	0	0	0	0	1	0	1
<i>i</i>	0	0	0	0	0	0	1	1	0



Adjacency matrix and adjacency list representations for the example graph.

Adjacency Matrix

- Given an adjacency matrix, we can decide in $\Theta(1)$ time whether two vertices are connected by an edge.
- We can also list all the neighbors of a vertex in $\Theta(|V|)$ time by scanning the row corresponding to that vertex
- This is optimal in the worst case, however if a vertex has few neighbors, we still need to examine every entry in the row to find them all
- Also, adjacency matrices require $\Theta(|V|^2)$ space, regardless of how many edges the graph has, so it is only space efficient for very *dense* graphs

Adjacency Lists

- For *sparse* graphs — graphs with relatively few edges — we're better off with adjacency lists
- An adjacency list is an array of linked lists, one list per vertex
- Each linked list stores the neighbors of the corresponding vertex

Adjacency Lists

- The total space required for an adjacency list is $O(|V| + |E|)$
- Listing all the neighbors of a node v takes $O(1 + \text{deg}(v))$ time
- We can determine if (u, v) is an edge in $O(1 + \text{deg}(u))$ time by scanning the neighbor list of u
- Note that we can speed things up by storing the neighbors of a node not in lists but rather in hash tables
- Then we can determine if an edge is in the graph in expected $O(1)$ time and still list all the neighbors of a node v in $O(1 + \text{deg}(v))$ time