CS 561, Lecture 2: Hash Tables, Skip Lists, Bloom Filters, Count-Min sketch

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Outline

• Hash Tables
• Skip Lists
• Count-Min Sketch
A dictionary ADT implements the following operations

- $\text{Insert}(x)$: puts the item $x$ into the dictionary
- $\text{Delete}(x)$: deletes the item $x$ from the dictionary
- $\text{IsIn}(x)$: returns true iff the item $x$ is in the dictionary
Frequently, we think of the items being stored in the dictionary as *keys*.

The keys typically have *records* associated with them which are carried around with the key but not used by the ADT implementation.

Thus we can implement functions like:

- \textit{Insert}(k,r): puts the item \((k,r)\) into the dictionary if the key \(k\) is not already there, otherwise returns an error.
- \textit{Delete}(k): deletes the item with key \(k\) from the dictionary.
- \textit{Lookup}(k): returns the item \((k,r)\) if \(k\) is in the dictionary, otherwise returns null.
Implementing Dictionaries

• The simplest way to implement a dictionary ADT is with a linked list
• Let \( l \) be a linked list data structure, assume we have the following operations defined for \( l \)
  – head(\( l \)): returns a pointer to the head of the list
  – next(p): given a pointer \( p \) into the list, returns a pointer to the next element in the list if such exists, null otherwise
  – previous(p): given a pointer \( p \) into the list, returns a pointer to the previous element in the list if such exists, null otherwise
  – key(p): given a pointer into the list, returns the key value of that item
  – record(p): given a pointer into the list, returns the record value of that item
Implement a dictionary with a linked list

- Q1: Write the operation Lookup(k) which returns a pointer to the item with key k if it is in the dictionary or null otherwise
- Q2: Write the operation Insert(k,r)
- Q3: Write the operation Delete(k)
- Q4: For a dictionary with $n$ elements, what is the runtime of all of these operations for the linked list data structure?
- Q5: Describe how you would use this dictionary ADT to count the number of occurrences of each word in an online book.
Dictionaries

• This linked list implementation of dictionaries is very slow
• Q: Can we do better?
• A: Yes, with hash tables, AVL trees, etc
Hash Tables

Hash Tables implement the Dictionary ADT, namely:

- Insert(x) - $O(1)$ expected time, $\Theta(n)$ worst case
- Lookup(x) - $O(1)$ expected time, $\Theta(n)$ worst case
- Delete(x) - $O(1)$ expected time, $\Theta(n)$ worst case
Direct Addressing

• Suppose universe of keys is $U = \{0, 1, \ldots, m - 1\}$, where $m$ is not too large
• Assume no two elements have the same key
• We use an array $T[0..m - 1]$ to store the keys
• Slot $k$ contains the elem with key $k$
Direct Address Functions

DA-Search(T,k){ return T[k];}
DA-Insert(T,x){ T[key(x)] = x;}
DA-Delete(T,x){ T[key(x)] = NIL;}

Each of these operations takes $O(1)$ time
Direct Addressing Problem

- If universe $U$ is large, storing the array $T$ may be impractical
- Also much space can be wasted in $T$ if number of objects stored is small
- Q: Can we do better?
- A: Yes we can trade time for space
“Key” Idea: An element with key $k$ is stored in slot $h(k)$, where $h$ is a hash function mapping $U$ into the set $\{0, \ldots, m-1\}$

Main problem: Two keys can now hash to the same slot

Q: How do we resolve this problem?

A1: Try to prevent it by hashing keys to “random” slots and making the table large enough

A2: Chaining

A3: Open Addressing
In chaining, all elements that hash to the same slot are put in a linked list.

CH-Insert(T,x){Insert x at the head of list T[h(key(x))];}
CH-Search(T,k){search for elem with key k in list T[h(k)];}
CH-Delete(T,x){delete x from the list T[h(key(x))];}
• CH-Insert and CH-Delete take $O(1)$ time if the list is doubly linked and there are no duplicate keys

• Q: How long does CH-Search take?

• A: It depends. In particular, depends on the load factor, $\alpha = n/m$ (i.e. average number of elems in a list)
• Worst case analysis: everyone hashes to one slot so $\Theta(n)$
• For average case, make the *simple uniform hashing* assumption: any given elem is equally likely to hash into any of the $m$ slots, indep. of the other elems
• Let $n_i$ be a random variable giving the length of the list at the $i$-th slot
• Then time to do a search for key $k$ is $1 + n_{h(k)}$
Q: What is $E(n_{h(k)})$?
A: We know that $h(k)$ is uniformly distributed among $\{0, \ldots, m-1\}$
Thus, $E(n_{h(k)}) = \sum_{i=0}^{m-1} (1/m)n_i = n/m = \alpha$
Hash Functions

- Want each key to be equally likely to hash to any of the $m$ slots, independently of the other keys.
- Key idea is to use the hash function to “break up” any patterns that might exist in the data.
- We will always assume a key is a natural number (can e.g. easily convert strings to natural numbers).
Division Method

- $h(k) = k \mod m$
- Want $m$ to be a \textit{prime number}, which is not too close to a power of 2
- Why? Reduces collisions in the case where there is periodicity in the keys inserted
Hash Tables implement the Dictionary ADT, namely:

- Insert(x) - $O(1)$ expected time, $\Theta(n)$ worst case
- Lookup(x) - $O(1)$ expected time, $\Theta(n)$ worst case
- Delete(x) - $O(1)$ expected time, $\Theta(n)$ worst case
Skip List

- Enables insertions and searches for ordered keys in $O(\log n)$ expected time
- Very elegant randomized data structure, simple to code but analysis is subtle
- They guarantee that, with high probability, all the major operations take $O(\log n)$ time (e.g. Find-Max, Predecessor/Successor)
- Can even enable “find-i-th value” if store with each edge the number of elements that edge skips
Skip List

- A skip list is basically a collection of doubly-linked lists, $L_1, L_2, \ldots, L_x$, for some integer $x$
- Each list has a special head and tail node, the keys of these nodes are assumed to be $-\text{MAXNUM}$ and $+\text{MAXNUM}$ respectively
- The keys in each list are in sorted order (non-decreasing)
- Every node is stored in the bottom list
- For each node in the bottom list, we flip a coin over and over until we get tails. For each heads, we make a duplicate of the node.
- The duplicates are stacked up in levels and the nodes on each level are strung together in sorted linked lists
- Each node $v$ stores a search key ($\text{key}(v)$), a pointer to its next lower copy ($\text{down}(v)$), and a pointer to the next node in its level ($\text{right}(v)$).
• To do a search for a key, $x$, we start at the leftmost node $L$ in the highest level
• We then scan through each level as far as we can without passing the target value $x$ and then proceed down to the next level
• The search ends either when we find the key $x$ or fail to find $x$ on the lowest level
Search

SkipListFind(x, L){
    v = L;
    while (v != NULL) and (Key(v) != x){
        if (Key(Right(v)) > x)
            v = Down(v);
        else
            v = Right(v);
    }
    return v;
}
Search Example
Insert

$p$ is a constant between 0 and 1, typically $p = 1/2$, let rand() return a random value between 0 and 1

Insert(k){
First call Search(k), let pLeft be the leftmost elem $\leq k$ in L_1
Insert k in L_1, to the right of pLeft
i = 2;
while (rand() <= p){
  insert k in the appropriate place in L_i;
}
}
Deletion

- Deletion is very simple
- First do a search for the key to be deleted
- Then delete that key from all the lists it appears in from the bottom up, making sure to “zip up” the lists after the deletion
Analysis

- Intuitively, each level of the skip list has about half the number of nodes of the previous level, so we expect the total number of levels to be about $O(\log n)$
- Similarly, each time we add another level, we cut the search time in half except for a constant overhead
- So after $O(\log n)$ levels, we would expect a search time of $O(\log n)$
- We will now formalize these two intuitive observations
• For some key, $i$, let $X_i$ be the maximum height of $i$ in the skip list.
• Q: What is the probability that $X_i \geq 2 \log n$?
• A: If $p = 1/2$, we have:

\[
P(X_i \geq 2 \log n) = \left( \frac{1}{2} \right)^{2 \log n}
\]

\[
= \frac{1}{(2^{\log n})^2}
\]

\[
= \frac{1}{n^2}
\]

• Thus the probability that a particular key $i$ achieves height $2 \log n$ is $\frac{1}{n^2}$
Q: What is the probability that any key achieves height $2 \log n$?
A: We want

$$P(X_1 \geq 2 \log n \text{ or } X_2 \geq 2 \log n \text{ or } \ldots \text{ or } X_n \geq 2 \log n)$$

By a Union Bound, this probability is no more than

$$P(X_1 \geq k \log n) + P(X_2 \geq k \log n) + \cdots + P(X_n \geq k \log n)$$

Which equals:

$$\sum_{i=1}^{n} \frac{1}{n^2} = \frac{n}{n^2} = 1/n$$
• This probability gets small as $n$ gets large
• In particular, the probability of having a skip list of size exceeding $2 \log n$ is $o(1)$
• If an event occurs with probability $1 - o(1)$, we say that it occurs with high probability
• Key Point: The height of a skip list is $O(\log n)$ with high probability.
A trick for computing expectations of discrete positive random variables:

- Let $X$ be a discrete r.v., that takes on values from 1 to $n$

$$E(X) = \sum_{i=1}^{n} P(X \geq i)$$
Why?

\[
\sum_{i=1}^{n} P(X \geq i) = P(X = 1) + P(X = 2) + P(X = 3) + \ldots \\
+ P(X = 2) + P(X = 3) + P(X = 4) + \ldots \\
+ P(X = 3) + P(X = 4) + P(X = 5) + \ldots \\
+ \ldots \\
= 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + 3 \cdot P(X = 3) + \ldots \\
= E(X)
\]
Q: How much memory do we expect a skip list to use up?

• Let $X_i$ be the number of lists that element $i$ is inserted in.
• Q: What is $P(X_i \geq 1)$, $P(X_i \geq 2)$, $P(X_i \geq 3)$?
• Q: What is $P(X_i \geq k)$ for general $k$?
• Q: What is $E(X_i)$?
• Q: Let $X = \sum_{i=1}^{n} X_i$. What is $E(X)$?
Search Time

- It's easier to analyze the search time if we imagine running the search backwards.
- Imagine that we start at the found node $v$ in the bottommost list and we trace the path backwards to the top leftmost sentinel, $L$.
- This will give us the length of the search path from $L$ to $v$, which is the time required to do the search.
Backwards Search

SLFback(v) {
    while (v != L) {
        if (Up(v) != NIL)
            v = Up(v);
        else
            v = Left(v);
    }
}
Backward Search

- For every node \( v \) in the skip list \( \text{Up}(v) \) exists with probability \( 1/2 \). So for purposes of analysis, \( \text{SLFBack} \) is the same as the following algorithm:

\[
\text{FlipWalk}(v)\
\quad\text{while} \ (v \neq L)\
\quad\quad\text{if} \ (\text{COINFLIP} == \text{HEADS})\
\quad\quad\quad v = \text{Up}(v);\
\quad\quad\quad \text{else}\
\quad\quad\quad v = \text{Left}(v);\
\quad\}\]
Analysis

- For this algorithm, the expected number of heads is exactly the same as the expected number of tails.
- Thus the expected run time of the algorithm is twice the expected number of upward jumps.
- Since we already know that the number of upward jumps is $O(\log n)$ with high probability, we can conclude that the expected search time is $O(\log n)$. 
Bloom Filters

- Randomized data structure for representing a set. Implements:
  - Insert(x):
  - IsMember(x):
- Allow false positives but require very little space
- Used frequently in: Databases, networking problems, p2p networks, packet routing
Bloom Filters

- Have $m$ slots, $k$ hash functions, $n$ elements; assume hash functions are all independent
- Each slot stores 1 bit, initially all bits are 0
- Insert($x$) : Set the bit in slots $h_1(x), h_2(x), ..., h_k(x)$ to 1
- IsMember($x$) : Return yes iff the bits in $h_1(x), h_2(x), ..., h_k(x)$ are all 1
$m$ slots, $k$ hash functions, $n$ elements; assume hash functions are all independent

Then $P(\text{fixed slot is still } 0) = (1 - 1/m)^{kn}$

Useful fact from Taylor expansion of $e^{-x}$:

$e^{-x} - x^2/2 \leq 1 - x \leq e^{-x}$ for $x < 1$

Then if $x \leq 1$

$$e^{-x}(1 - x^2) \leq 1 - x \leq e^{-x}$$
Analysis

• Thus we have the following to good approximation.

\[ Pr(\text{fixed slot is still 0}) = (1 - \frac{1}{m})^{kn} \approx e^{-kn/m} \]

• Let \( p = e^{-kn/m} \) and let \( \rho \) be the fraction of 0 bits after \( n \) elements inserted then

\[ Pr(\text{false positive}) = (1 - \rho)^k \approx (1 - p)^k \]

• Where the first approximation holds because \( \rho \) is very close to \( p \) (by a Martingale argument beyond the scope of this class)
Analysis

- Want to minimize \((1 − p)^k\), which is equivalent to minimizing 
  \[ g = k \ln(1 − p) \]
- Trick: Note that 
  \[ g = −(m/n) \ln(p) \ln(1 − p) \]
- By symmetry, this is minimized when \(p = 1/2\) or equivalently 
  \[ k = (m/n) \ln 2 \]
- False positive rate is then 
  \[ (1/2)^k \approx (0.6185)^{m/n} \]
Tricks

- Can get the union of two sets by just taking the bitwise-or of the bit-vectors for the corresponding Bloom filters.
- Can easily half the size of a bloom filter - assume size is power of 2 then just bitwise-or the first and second halves together.
- Can approximate the size of the intersection of two sets - inner product of the bit vectors associated with the Bloom filters is a good approximation to this.
Extensions

- Bloomier Filters: Also allow for data to be inserted in the filter - similar functionality to hash tables but less space, and the possibility of false positives.
Data Streams

- A router forwards packets through a network
- A natural question for an administrator to ask is: what is the list of substrings of a fixed length that have passed through the router more than a predetermined threshold number of times
- This would be a natural way to try to, for example, identify worms and spam
- Problem: the number of packets passing through the router is *much* too high to be able to store counts for every substring that is seen!
Data Streams

- This problem motivates the data stream model
- Informally: there is a stream of data given as input to the algorithm
- The algorithm can take at most one pass over this data and must process it sequentially
- The memory available to the algorithm is much less than the size of the stream
- In general, we won’t be able to solve problems exactly in this model, only approximate
Our Problem

- We are presented with a stream of items $i$
- We want to get a good approximation to the value $\text{Count}(i,T)$, which is the number of times we have seen item $i$ up to time $T$
Count-Min Sketch

- Our solution will be to use a data structure called a *Count-Min Sketch*
- This is a randomized data structure that will keep approximate values of \( \text{Count}(i,T) \)
- It is implemented using \( k \) hash functions and \( m \) counters
Count-Min Sketch

- Think of our $m$ counters as being in a 2-dimensional array, with $m/k$ counters per row and $k$ rows
- Let $C_{a,b}$ be the counter in row $a$ and column $b$
- Our hash functions map items from the universe into counters
- In particular, hash function $h_a$ maps item $i$ to counter $C_{a,h_a(i)}$
Updates

- Initially all counters are set to 0
- When we see item $i$ in the data stream we do the following
- For each $1 \leq a \leq k$, increment $C_{a,h_a(i)}$
Count Approximations

- Let $C_{a,b}(T)$ be the value of the counter $C_{a,b}$ after processing $T$ tuples
- We approximate Count($i,T$) by returning the value of the smallest counter associated with $i$
- Let $m(i,T)$ be this value
Main Theorem:

- For any item $i$, $m(i, T) \geq \text{Count}(i, T)$
- With probability at least $1 - e^{-m\epsilon/e}$ the following holds: $m(i, T) \leq \text{Count}(i, T) + \epsilon T$
Proof

- Easy to see that $m(i, T) \geq \text{Count}(i, T)$, since each counter $C_{a, h_a(i)}$ incremented by $c_t$ every time pair $(i, c_t)$ is seen.
- Hard Part: Showing $m(i, T) \leq \text{Count}(i, T) + \epsilon T$.
- To see this, we will first consider the specific counter $C_{1, h_1(i)}$ and then use symmetry.
Proof

• Let $Z_1$ be a random variable giving the amount the counter is incremented by items other than $i$
• Let $X_t$ be an indicator r.v. that is 1 if $j$ is the $t$-th item, and $j \neq i$ and $h_1(i) = h_1(j)$
• Then $Z_1 = \sum_{t=1}^{T} X_t$
• But if the hash functions are “good”, then if $i \neq j$, $Pr(h_1(i) = h_1(j)) \leq k/m$ (specifically, we need the hash functions to come from a 2-universal family, but we won’t get into that in this class)
• Hence, $E(X_t) \leq k/m$
• Thus, by linearity of expectation, we have that:

\[ E(Z_1) = \sum_{t=1}^{T} \left( \frac{k}{m} \right) \leq Tk/m \]  

(1)

(2)

• We now need to make use of a very important inequality: Markov’s inequality
Markov’s Inequality

- Let $X$ be a random variable that only takes on non-negative values
- Then for any $\lambda > 0$:

$$\Pr(X \geq \lambda) \leq E(X)/\lambda$$

- Proof of Markov’s: Assume instead that there exists a $\lambda$ such that $\Pr(X \geq \lambda)$ was actually larger than $E(X)/\lambda$
- But then the expected value of $X$ would be at least $\lambda \cdot \Pr(X \geq \lambda) > E(X)$, which is a contradiction!!!
Proof

- Now, by Markov’s inequality,

\[ Pr(Z_1 \geq \epsilon T') \leq \frac{(Tk/m)}{(\epsilon T')} = k/(m\epsilon) \]

- This is the event where \( Z_1 \) is “bad” for item \( i \).
Proof (Cont’d)

- Now again assume our $k$ hash functions are “good” in the sense that they are independent.
- Then we have that

\[
\prod_{i=1}^{k} \Pr(Z_j \geq \epsilon T) \leq \left(\frac{k}{m \epsilon}\right)^k
\]
Proof

• Finally, we want to choose a $k$ that minimizes this probability.
• Using calculus, we can see that the probability is minimized when $k = \frac{m\epsilon}{e}$, in which case

$$\left(\frac{k}{m\epsilon}\right)^k = e^{-m\epsilon/e}$$

• This completes the proof!
Recap

- Our Count-Min Sketch is very good at giving estimating counts of items with very little external space.
- Tradeoff is that it only provides approximate counts, but we can bound the approximation!
- Note: Can use the Count-Min Sketch to keep track of all the items in the stream that occur more than a given threshold ("heavy hitters")
- Basic idea is to store an item in a list of "heavy hitters" if its count estimate ever exceeds some given threshold.