University of New Mexico
Department of Computer Science

Final Examination

CS 561 Data Structures and Algorithms
Fall, 2018

Name:
Email:

- This exam lasts 120 minutes. It is closed book and notes, and no electronic devices are permitted. However, you are allowed to use 2 pages of handwritten "cheat sheets"
- *Show your work!* You will not get full credit if we cannot figure out how you arrived at your answer.
- Write your solution in the space provided for the corresponding problem.
- If any question is unclear, ask for clarification.

Question	Points	Score	Grader
1	20		
2	20		
3	20		
4	20		
5	20		
Total	100		

1. **Short Answer (2 points each)**

   Answer the following questions using the *simplest possible theta* notation. Assume as usual, that $f(n)$ is $\theta(1)$ for constant values of n.

   (a) Solution to the recurrence $T(n) = T(n-1) + n$

   (b) Solution to the following recurrence $T(n) = 2T(n/2) + \sqrt{n}$

   (c) Solution to the following recurrence relation: $f(n) = 4f(n-1) - 3f(n-2)$.

   (d) Expected time to find the minimum item in a skip list with n items?

   (e) Assume an operation, FOO, has amortized cost $O(1)$ but worst case cost $O(n)$. What is the worst case time for n total calls to FOO?

(f) Runtime of fastest algorithm to compute the shortest path between two nodes in a weighted graph with n nodes and m edges, when the graph does not have negative edges?

(g) Runtime of fastest algorithm to find whether a weighted graph, with negative edges, has a negative cycle?

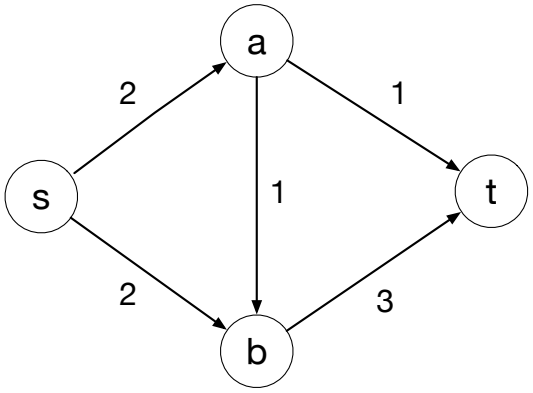
(h) You throw n items into n bins independently and uniformly at random. What is the expected number of pairs of items where both items in the pair fall in the same bin?

(i) Using Markov's inequality in the last problem, what is an upper bound on the probability that there are $\theta(n^2)$ pairs of items where both items in the pair fall in the same bin?

(j) You throw n items into n^4 bins independently and uniformly at random. Using a union bound, what is an upper bound on the probability that *any* pair of items fall in the same bin?

2. **Induction and Linear Programming**

(a) (10 points) Give a linear program to find the maximum flow that can be sent from s to t in this graph. The numbers on each edge represents the capacity of that edge.
Hint: For each edge $u \to v$, let $f_{u \to v}$ be the flow over that edge.



(b) (10 points) Prove via induction that any 3-regular graph can be colored with at most 4 colors. Recall that a 3-regular graph is one where every node has degree 3. A *coloring* of a graph G is an assignment of a color to each node in G such that the endpoints of each edge in G are assigned different colors. Don't forget to include BC, IH and IS in your proof.

Hint: Perform induction on, n, the number of nodes in G. In the IS, think about how to make G smaller, so that you can use the IH.

3. **SENSOR-PLACEMENT**

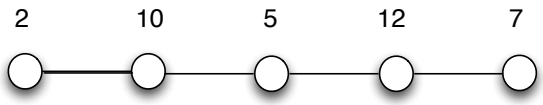
In the SENSOR-PLACEMENT problem, you are given a graph $G = (V, E)$ that represents a sensor network. The nodes of V represent sensors, and each node $v \in V$ has a positive weight $w(v)$ that indicates the importance of the data collected by that sensor. There is an edge $(u, v) \in E$ if sensors u and v are neighbors in the network, and in such a case, u and v can not both be turned on because of interference issues.

Your goal is to find a subset $S \subseteq V$ such that (1) no two nodes in S are neighbors; and (2) the sum of the weights of all nodes in S is maximized among all such sets.

(a) (6 points) Show that SENSOR-PLACEMENT is NP-Hard by a reduction from one of the following: 3-SAT, CLIQUE, INDEPENDENT SET, VERTEX COVER, 3-COLORABLE or HAMILTONIAN CYCLE.

(b) (4 points) Now your boss decides to create a new easier problem by assuming that G is a path (see figure). He claims that the following greedy algorithm will find an optimal set S for this new problem. GREEDY initially starts with an empty set S. Then it repeats the following until the set V is empty: (1) find a vertex $v \in V$ of maximum weight; (2) add v to S; and (3) remove all neighbors of v from V.

Show your boss is wrong by giving a problem instance, i.e. a graph G that is a path, and weights for the vertices of G, for which GREEDY does not return the optimal solution.



(c) (6 points) Now describe how to solve this new problem (where G is a path) using dynamic programming. Let the vertices in the path have labels $v_1, \ldots, v_n$. Let $m(i)$ be the max weight of the set S that is obtainable by considering only vertices v_1 through v_i. First give the recurrence relation (and base case(s)) for $m(i)$ below.

(d) (4 points) Now describe how to create a dynamic program to find S. What is the run time of your algorithm as a function of n?

4. **Fitting a Line**

In this problem, you want to use gradient descent to find the best line that fits a collection of data. You have a collection of n data items, where for $1 \leq i \leq n$, item i is a tuple (u_i, v_i). Your goal is to find a a line $mx + b$ that minimizes

$$\sum_{i=1}^{n}((mu_i + b) - v_i)^2$$

Assume that m and b are both in the range -10 to 10.

(a) (3 points) In one sentence, describe the convex search space κ (dimensions and boundaries)?

(b) (3 points) What is D, the diameter of κ? You can leave square roots in your answer.

(c) (2 points) What is the function $f(m, b)$ that you are minimizing?

(d) (4 points) What is the gradient of $f(m, b)$?

(e) (4 points) Assume that for all $1 \leq i \leq n$, u_i and v_i are both in the range -10 to 10. What is G, the max norm of ∇f as a function of n? Feel free to leave squares and square roots in your answer.

(f) (4 points) Your boss wants to speed up the algorithm. She suggests at each iteration of gradient descent to choose a random i between 1 and n, set $f_i(m, b) = ((mu_i + b) - v_i)^2$, and then update based on the gradient of $f_i(m, b)$ rather than the gradient of $f(m, b)$. Will this work? Justify your answer in one sentence.

5. **UNIQUE-SET-COVER**

Consider the following problem called UNIQUE-SET-COVER. The input is an n-item set S, and a collection of m subsets $S_1, S_2, \ldots S_m \subseteq S$, where each item of S is in k of the subsets. Our goal is to find a collection of subsets that maximizes the number of items that are *uniquely covered*, where an item is uniquely covered if it is in exactly one subset in the collection.

(a) (6 points) Fix a real number p between 0 and 1 and consider the following algorithm: **For each index i, select set S_i independently with probability p.**
What is the expected number of items that are uniquely covered by this algorithm? Express your answer as a function of p and k.

(b) (6 points) What value of p maximizes this expectation?

(c) (8 points) Now describe a polynomial-time randomized algorithm for UNIQUE-SET-COVER. Prove that its expected approximation ratio is $O(1)$. Hint: Use Bernoulli's inequality which says that for all $t \geq -1$ and $n \geq 1$, $(1+t)^n \geq 1 + nt$.