CS 561, Randomized Algorithms

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Quicksort

- Based on divide and conquer strategy
- Worst case is $\Theta(n^2)$
- Expected running time is $\Theta(n \log n)$
- An In-place sorting algorithm
- Almost always the fastest sorting algorithm

Quicksort

- **Divide:** Pick some element A[q] of the array A and partition A into two arrays A_1 and A_2 such that every element in A_1 is $\leq$ A[q], and every element in A_2 is $>$ A[p]
- **Conquer:** Recursively sort A_1 and A_2
- **Combine:** A_1 concatenated with $A[q]$ concatenated with A_2 is now the sorted version of A

The Algorithm

```
//PRE: A is the array to be sorted, p>=1;
//     r is <= the size of A
//POST: A[p..r] is in sorted order
Quicksort (A,p,r){
  if (p<r){
    q = Partition (A,p,r);
    Quicksort (A,p,q-1);
    Quicksort (A,q+1,r);
}
```

Partition

```
//PRE: A[p..r] is the array to be partitioned, p>=1 and r <= size
//     of A, A[r] is the pivot element
//POST: Let A' be the array A after the function is run.  Then
//      A'[p..r] contains the same elements as A[p..r].  Further,
//      all elements in A'[p..res-1] are <= A[r], A'[res] = A[r],
//      and all elements in A'[res+1..r] are > A[r]
Partition (A,p,r){
  x = A[r];
  i = p-1;
  for (j=p;j<=r-1;j++){
    if (A[j]<=x){
      i++;
      exchange A[i] and A[j];
  }}
  exchange A[i+1] and A[r];
  return i+1;
}
```

Correctness

Basic idea: The array is partitioned into four regions, x is the pivot

- Region 1: Region that is less than or equal to x
  (between p and i)
- Region 2: Region that is greater than x
  (between $i + 1$ and $j - 1$)
- Region 3: Unprocessed region
  (between j and $r - 1$)
- Region 4: Region that contains x only
  (r)

Region 1 and 2 are growing and Region 3 is shrinking

Loop Invariant

At the beginning of each iteration of the for loop, for any index k:

1. If $p \leq k \leq i$ then $A[k] \leq x$
2. If $i + 1 \leq k \leq j - 1$ then $A[k] > x$
3. If $k = r$ then $A[k] = x$

Example

- Consider the array (2 6 4 1 5 3)

At-Home Exercise (Soln on p. 147)

- Show this invariant holds before the loop begins (Initialization)
- Show if the invariant holds after the $i-1$-th iteration, that it will hold after the i-th iteration (Maintenance)
- Show that if the invariant holds when the loop exits, that the array will be successfully partitioned (Termination)

Analysis

- The function Partition takes $O(n)$ time. Why?

Randomized Quick-Sort

- We'd like to ensure that we get reasonably good splits reasonably quickly
- Q: How do we ensure that we "usually" get good splits? How can we ensure this even for worst case inputs?
- A: We use randomization.

R-Partition

```
//PRE: A[p..r] is the array to be partitioned, p>=1 and r <= size
//     of A
//POST: Let A' be the array A after the function is run.  Then
//      A'[p..r] contains the same elements as A[p..r].  Further,
//      all elements in A'[p..res-1] are <= A[i], A'[res] = A[i],
//      and all elements in A'[res+1..r] are > A[i], where i is
//      a random number between $p$ and $r$.
R-Partition (A,p,r){
  i = Random(p,r);
  exchange A[r] and A[i];
  return Partition(A,p,r);
}
```

Randomized Quicksort

```
//PRE: A is the array to be sorted, p>=1,  and r is <= the size of A
//POST: A[p..r] is in sorted order
R-Quicksort (A,p,r){
  if (p<r){
    q = R-Partition (A,p,r);
    R-Quicksort (A,p,q-1);
    R-Quicksort (A,q+1,r);
}
```

Analysis

- R-Quicksort is a *randomized* algorithm
- The run time is a *random variable*
- We'd like to analyze the *expected* run time of R-Quicksort
- To do this, we first need to learn some basic probability theory.

Probability Definitions

(from Appendix C.3)

- A *random variable* is a variable that takes on one of several values, each with some probability. (Example: if X is the outcome of the roll of a die, X is a random variable)
- The *expected value* of a random variable,X is defined as:

$$E(X) = \sum_x x * P(X = x)$$

(Example if X is the outcome of the roll of a three sided die,

$$\begin{aligned} E(X) &= 1 * (1/3) + 2 * (1/3) + 3 * (1/3) \\ &= 2 \end{aligned}$$

Probability Definitions

- Two events A and B are *mutually exclusive* if $A \bigcap B$ is the empty set (Example: A is the event that the outcome of a die is 1 and B is the event that the outcome of a die is 2)
- Two random variables X and Y are *independent* if for all x and y, $P(X = x \text{ and } Y = y) = P(X = x)P(Y = y)$ (Example: let X be the outcome of the first roll of a die, and Y be the outcome of the second roll of the die. Then X and Y are independent.)

Probability Definitions

- An *Indicator Random Variable* associated with event A is defined as:
  - $I(A) = 1$ if A occurs
  - $I(A) = 0$ if A does not occur
- Example: Let A be the event that the roll of a die comes up 2. Then $I(A)$ is 1 if the die comes up 2 and 0 otherwise.

Linearity of Expectation

- Let X and Y be two random variables
- Then $E(X+Y) = E(X) + E(Y)$
- (Holds even if X and Y are not independent.)

- More generally, let $X_1, X_2, \ldots, X_n$ be n random variables
- Then

$$E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$$

Example

- For $1 \leq i \leq n$, let X_i be the outcome of the i-th roll of three-sided die
- Then

$$E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = 2n$$

Example

- Indicator Random Variables and Linearity of Expectation used together are a very powerful tool
- The “Birthday Paradox” illustrates this point
- To analyze the run time of quicksort, we will also use indicator r.v.’s and linearity of expectation (analysis will be similar to “birthday paradox” problem)

“Birthday Paradox”

- Assume there are k people in a room, and n days in a year
- Assume that each of these k people is born on a day chosen uniformly at random from the n days
- Q: What is the expected number of pairs of individuals that have the same birthday?
- We can use indicator random variables and linearity of expectation to compute this

Analysis

- For all $1 \leq i < j \leq k$, let $X_{i,j}$ be an indicator random variable defined such that:
  - $X_{i,j} = 1$ if person i and person j have the same birthday
  - $X_{i,j} = 0$ otherwise
- Note that for all i, j,

$$
\begin{aligned}
E(X_{i,j}) &= P(\text{person i and j have same birthday}) \\
&= 1/n
\end{aligned}
$$

Analysis

- Let X be a random variable giving the number of pairs of people with the same birthday
- We want $E(X)$
- Then $X = \sum_{(i,j)} X_{i,j}$
- So $E(X) = E(\sum_{(i,j)} X_{i,j})$

Analysis

$$
\begin{aligned}
E(X) &= E(\sum_{(i,j)} X_{i,j}) \\
&= \sum_{(i,j)} E(X_{i,j}) \\
&= \sum_{(i,j)} 1/n \\
&= \binom{k}{2} 1/n \\
&= \frac{k(k-1)}{2n}
\end{aligned}
$$

The second step follows by Linearity of Expectation

Reality Check

- Thus, if $k(k-1) \geq 2n$, expected number of pairs of people with same birthday is at least 1
- Thus if have at least $\sqrt{2n}+1$ people in the room, can expect to have at least two with same birthday
- For $n = 365$, if $k = 28$, expected number of pairs with same birthday is 1.04

In-Class Exercise

- Assume there are k people in a room, and n days in a year
- Assume that each of these k people is born on a day chosen uniformly at random from the n days
- Let X be the number of groups of *three* people who all have the same birthday. What is $E(X)$?
- Let $X_{i,j,k}$ be an indicator r.v. which is 1 if people i,j, and k have the same birthday and 0 otherwise

In-Class Exercise

- Q1: Write the expected value of X as a function of the $X_{i,j,k}$ (use linearity of expectation)
- Q2: What is $E(X_{i,j,k})$?
- Q3: What is the total number of groups of three people out of k?
- Q4: What is $E(X)$?

Plan of Attack

"If you get hold of the head of a snake, the rest of it is mere rope" - Akan Proverb

- We will analyze the *total* number of comparisons made by quicksort
- We will let X be the total number of comparisons made by R-Quicksort
- We will write X as the sum of a bunch of indicator random variables
- We will use linearity of expectation to compute the expected value of X

Notation

- Let A be the array to be sorted
- Let z_i be the i-th smallest element in the array A
- Let $Z_{i,j} = \{z_i, z_{i+1}, \ldots, z_j\}$

Indicator Random Variables

- Let $X_{i,j}$ be 1 if z_i is compared with z_j and 0 otherwise
- Note that $X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}$
- Further note that

$$E(X) = E(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{i,j})$$

Questions

- Q1: So what is $E(X_{i,j})$?
- A1: It is $P(z_i$ is compared to $z_j)$
- Q2: What is $P(z_i$ is compared to $z_j)$?
- A2: It is:

  P(either z_i or z_j are the first elems in $Z_{i,j}$ chosen as pivots)

- Why?
  - If no element in $Z_{i,j}$ has been chosen yet, no two elements in $Z_{i,j}$ have yet been compared, and all of $Z_{i,j}$ is in same list
  - If some element in $Z_{i,j}$ other than z_i or z_j is chosen first, z_i and z_j will be split into separate lists (and hence will never be compared)

More Questions

- Q: What is

  $$P(\text{either } z_i \text{ or } z_j \text{ are first elems in } Z_{i,j} \text{ chosen as pivots})$$

- A: $P(z_i$ chosen as first elem in $Z_{i,j}) +$ $P(z_j$ chosen as first elem in $Z_{i,j})$
- Further note that number of elems in $Z_{i,j}$ is $j - i + 1$, so

  $$P(z_i \text{ chosen as first elem in } Z_{i,j}) = \frac{1}{j-i+1}$$

  and

  $$P(z_j \text{ chosen as first elem in } Z_{i,j}) = \frac{1}{j-i+1}$$

- Hence

$$P(z_i \text{ or } z_j \text{ are first elems in } Z_{i,j} \text{ chosen as pivots}) = \frac{2}{j-i+1}$$

Conclusion

$$E(X_{i,j}) = P(z_i \text{ is compared to } z_j) \quad (1)$$

$$= \frac{2}{j - i + 1} \quad (2)$$

Putting it together

$$
\begin{aligned}
E(X) &= E(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}) && (3)\\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E(X_{i,j}) && (4)\\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} && (5)\\
&= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} && (6)\\
&< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} && (7)\\
&= \sum_{i=1}^{n-1} O(\log n) && (8)\\
&= O(n \log n) && (9)
\end{aligned}
$$

Questions

- Q: Why is $\sum_{k=1}^{n} \frac{2}{k} = O(\log n)$?
- A:

$$\begin{aligned} \sum_{k=1}^{n} \frac{2}{k} &= 2 \sum_{k=1}^{n} 1/k && (10) \\ &\leq 2(\ln n + 1) && (11) \end{aligned}$$

- Where the last step follows by an integral bound on the sum (p. 1067)

How Fast Can We Sort?

- Q: What is a lowerbound on the runtime of any sorting algorithm?
- We know that $\Omega(n)$ is a trivial lowerbound
- But all the algorithms we've seen so far are $O(n \log n)$ (or $O(n^2)$), so is $\Omega(n \log n)$ a lowerbound?

Comparison Sorts

- Definition: An sorting algorithm is a *comparison sort* if the sorted order they determine is based only on comparisons between input elements.
- Heapsort, mergesort, quicksort, bubblesort, and insertion sort are all comparison sorts
- We will show that any comparison sort must take $\Omega(n \log n)$

Comparisons

- Assume we have an input sequence $A = (a_1, a_2, \ldots, a_n)$
- In a comparison sort, we only perform tests of the form $a_i < a_j$, $a_i \leq a_j$, $a_i = a_j$, $a_i \geq a_j$, or $a_i > a_j$ to determine the relative order of all elements in A
- We'll assume that all elements are distinct, and so note that the only comparison we need to make is $a_i \leq a_j$.
- This comparison gives us a yes or no answer

Decision Tree Model

- A decision tree is a full binary tree that gives the possible sequences of comparisons made for a particular input array, A
- Each internal node is labelled with the indices of the two elements to be compared
- Each leaf node gives a permutation of A

Decision Tree Model

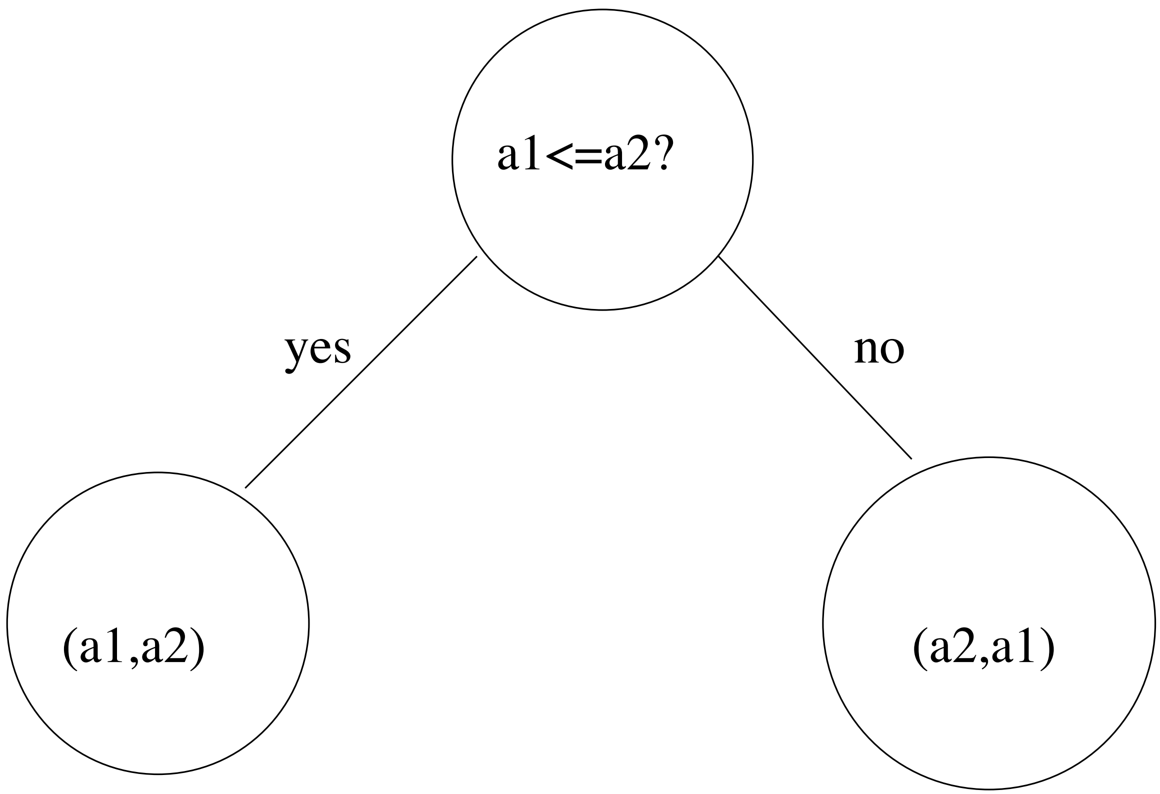
- The execution of the sorting algorithm corresponds to a path from the root node to a leaf node in the tree.
- We take the left child of the node if the comparison is $\leq$ and we take the right child if the comparison is $>$
- The internal nodes along this path give the comparisons made by the alg, and the leaf node gives the output of the sorting algorithm.

Leaf Nodes

- Any correct sorting algorithm must be able to produce each possible permutation of the input
- Thus there must be at least $n!$ leaf nodes
- The length of the longest path from the root node to a leaf in this tree gives the worst case run time of the algorithm (i.e. the height of the tree gives the worst case runtime)

Example

- Consider the problem of sorting an array of size two: $A = (a_1, a_2)$
- Following is a decision tree for this problem.



In-Class Exercise

- Give a decision tree for sorting an array of size three: $A = (a_1, a_2, a_3)$
- What is the height? What is the number of leaf nodes?

Height of Decision Tree

- Q: What is the height of a binary tree with at least $n!$ leaf nodes?
- A: If h is the height, we know that $2^h \geq n!$
- Taking log of both sides, we get $h \geq \log(n!)$

Height of Decision Tree

- Q: What is $\log(n!)$?
- A: It is

$$\begin{aligned}\log(n*(n-1)*\cdots*1) &= \log n + \log(n-1) + \cdots + \log 1\\ &\geq (n/2)\log(n/2)\\ &\geq (n/2)(\log n - \log 2)\\ &= \Omega(n \log n)\end{aligned}$$

- Thus any decision tree for sorting n elements will have a height of $\Omega(n \log n)$

Take Away

- We've just proven that any comparison-based sorting algorithm takes $\Omega(n \log n)$ time
- This does *not* mean that *all* sorting algorithms take $\Omega(n \log n)$ time
- In fact, there are non comparison-based sorting algorithms which, under certain circumstances, are asymptotically faster.

Bucket Sort

- Bucket sort assumes that the input is drawn from a uniform distribution over the range $[0, 1)$
- Basic idea is to divide the interval $[0, 1)$ into n equal size regions, or buckets
- We expect that a small number of elements in A will fall into each bucket
- To get the output, we can sort the numbers in each bucket and just output the sorted buckets in order

Bucket Sort

```
//PRE: A is the array to be sorted, all elements in A[i] are between
0 and 1 inclusive.
//POST: returns a list which is the elements of A in sorted order
BucketSort(A){
B = new List[]
n = length(A)
for (i=1;i<=n;i++){
  insert A[i] at end of list B[floor(n*A[i])];
}
for (i=0;i<=n-1;i++){
  sort list B[i] with insertion sort;
}
return the concatenated list B[0],B[1],...,B[n-1];
}
```

Bucket Sort

- Claim: If the input numbers are distributed uniformly over the range $[0, 1)$, then Bucket sort takes expected time $O(n)$
- Let $T(n)$ be the run time of bucket sort on a list of size n
- Let n_i be the random variable giving the number of elements in bucket $B[i]$
- Then $T(n) = \Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)$

Analysis

- We know $T(n) = \Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)$
- Taking expectation of both sides, we have

$$\begin{aligned} E(T(n)) &= E(\Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)) \\ &= \Theta(n) + \sum_{i=0}^{n-1} E(O(n_i^2)) \\ &= \Theta(n) + \sum_{i=0}^{n-1} (O(E(n_i^2))) \end{aligned}$$

- The second step follows by linearity of expectation
- The last step holds since for any constant a and random variable X, $E(aX) = aE(X)$ (see Equation C.21 in the text)

Analysis

- We claim that $E(n_i^2) = 2 - 1/n$
- To prove this, we define indicator random variables: $X_{ij} = 1$ if $A[j]$ falls in bucket i and 0 otherwise (defined for all i, $0 \le i \le n-1$ and j, $1 \le j \le n$)
- Thus, $n_i = \sum_{j=1}^{n} X_{ij}$
- We can now compute $E(n_i^2)$ by expanding the square and regrouping terms

Analysis

$$
\begin{aligned}
E(n_i^2) &= E((\sum_{j=1}^{n} X_{ij})^2) \\
&= E(\sum_{j=1}^{n} \sum_{k=1}^{n} X_{ij} X_{ik}) \\
&= E(\sum_{j=1}^{n} X_{ij}^2 + \sum_{1 \le j \le n} \sum_{1 \le k \le n, k \ne j} X_{ij} X_{ik}) \\
&= \sum_{j=1}^{n} E(X_{ij}^2) + \sum_{1 \le j \le n} \sum_{1 \le k \le n, k \ne j} E(X_{ij} X_{ik}))
\end{aligned}
$$

Analysis

- We can evaluate the two summations separately. X_{ij} is 1 with probability $1/n$ and 0 otherwise
- Thus $E(X_{ij}^2) = 1 * (1/n) + 0 * (1 - 1/n) = 1/n$
- Where $k \neq j$, the random variables X_{ij} and X_{ik} are independent
- For any two *independent* random variables X and Y, $E(XY) = E(X)E(Y)$ (see C.3 in the book for a proof of this)
- Thus we have that

$$
\begin{aligned}
E(X_{ij}X_{ik}) &= E(X_{ij})E(X_{ik}) \\
&= (1/n)(1/n) \\
&= (1/n^2)
\end{aligned}
$$

Analysis

- Substituting these two expected values back into our main equation, we get:

$$\begin{aligned}
E(n_i^2) &= \sum_{j=1}^{n} E(X_{ij}^2) + \sum_{1\le j\le n} \sum_{1\le k\le n, k\ne j} E(X_{ij}X_{ik})) \\
&= \sum_{j=1}^{n} (1/n) + \sum_{1\le j\le n} \sum_{1\le k\le n, k\ne j} (1/n^2) \\
&= n(1/n) + (n)(n-1)(1/n^2) \\
&= 1 + (n-1)/n \\
&= 2 - (1/n)
\end{aligned}$$

Analysis

- Recall that $E(T(n)) = \Theta(n) + \sum_{i=0}^{n-1}(O(E(n_i^2)))$
- We can now plug in the equation $E(n_i^2) = 2 - (1/n)$ to get

$$
\begin{aligned}
E(T(n)) &= \Theta(n) + \sum_{i=0}^{n-1} 2 - (1/n) \\
&= \Theta(n) + \Theta(n) \\
&= \Theta(n)
\end{aligned}
$$

- Thus the entire bucket sort algorithm runs in expected linear time