CS 561, Approximation Algorithms

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Outline

- SET-COVER
- MAX-SAT

SET-COVER

- Given a universe of elements $U = \{1, \ldots m\}$, and a family of subsets of U called $\mathbb{S}$
- For every $S \in \mathbb{S}$, there is a weight w_S
- Goal: Find a cover $C \subseteq \mathbb{S}$ of minimum weight $\sum_{S \in C} w_S$.
- A set C is a *cover*, if for all $i \in U$, there is a set $S \in C$ such that $i \in S$.

SET-COVER

- SET-COVER is NP-HARD (to show, reduce from VERTEX-COVER)
- Want to solve this problem frequently in e.g. computational biology
- There is an interesting approximation algorithm for it though
- IDEA: Solve an LP; Use the setting in the solution to assign probabilities to indicator rv's; Round these rv's

The Integer Program (IP)

Minimize: $\sum_{S \in \mathbb{S}} w_S x_S$

Subject to:

$\sum_{S: i \in S} x_S \geq 1$, $\forall i \in U$

$x_S \in \{0, 1\}$, $\forall S \in \mathbb{S}$

The Linear Program (LP)

Minimize: $\sum_{S \in \mathbb{S}} w_S x_S$

Subject to:

$\sum_{S: i \in S} x_S \geq 1$, $\forall i \in U$

$0 \leq x_S \leq 1$, $\forall S \in \mathbb{S}$

Analysis

- IDEA: Solve this LP in polynomial time
- PROBLEM: It gives us $x_S \in [0, 1]$ for all S. How do we decide whether to choose each set?
- IDEA: Choose set S with **probability** x_S

Example

- $U = \{a, b, c\}$
- $S_1 = \{a, b\}$; $S_2 = \{a, c\}$; $S_3 = \{b, c\}$
- $w_S = 1$ for all sets S

Example

- $U = \{a, b, c\}$
- $S_1 = \{a, b\}$; $S_2 = \{a, c\}$; $S_3 = \{b, c\}$
- $w_S = 1$ for all sets S

- LP Solution: $x_1^* = x_2^* = x_3^* = 1/2$
- Let R be the sets in the rounding
- Example Rounding: $R = \{S_1, S_2\}$
- Success! This gives a cover with optimal weight

Fact 1: Expected weight of R is no more than expected weight of OPT.

- Proof: For each possible set S, let X_S be an indicator r.v. that is 1 iff $S \in R$. Then we have

$$E\left(\sum_{S \in R} w_S\right) = E\left(\sum_S w_S X_S\right) = \sum_S w_S E(X_S) = \sum_S w_S x_S^*$$

- The last term is the weight of the LP solution which is at most the weight of the optimal solution.

Fact 2: Every element $i \in U$ is covered by R with probability at least $1 - 1/e$

- Proof: Fix an element $i \in U$. Let T be the sets in $\mathbb{S}$ that contain i. Then

$$
\begin{aligned}
Pr(i \text{ is not covered by } R) &= \prod_{S \in T} Pr(S \notin R) \\
&= \prod_{S \in T} (1 - x_S^*) \\
&\leq \prod_{S \in T} e^{-x_S^*} \\
&= e^{-\sum_{S \in T} x_S^*} \\
&\leq e^{-1}.
\end{aligned}
$$

Problem: May not always get a cover

- Problem: Each item covered with probability $1 - 1/e$, but likely that *some* item not covered.
- Idea: Round multiple times to get a cover with high probability.
- Increases the weight, but only by a logarithmic amount

Algorithm 1

1. Let x^* be a solution to the relaxed LP
2. For $t = 1$ to $2 \ln m$ do
   (a) Add each set S to R_t with probability x^*_S independently
3. Return $\bigcup_t R_t$

Analysis

Theorem 1: In one run with probability 1/4, Algorithm 1 (1) returns a cover, (2) with total weight at most $4 \ln m \cdot OPT$.

Proof: (1) For a fixed i, By Fact 1 and independence, we have

$$Pr(\text{i not covered}) \leq e^{-2 \ln m} = m^{-2}$$

Thus, by a union bound:

$$Pr(\text{any of the } m \text{ elements uncovered}) \leq m^{-1} \leq \frac{1}{4}.$$

(2) By Fact 2, expected weight of sets added in one iteration of the for loop is at most OPT. By linearity, expected weight over $2 \ln m$ iterations is at most $2 \ln m \cdot OPT$. Let W be the weight of the sets returned by the algorithm. By Markov's inequality, $Pr(W \geq 2E(W)) \leq 1/2$.

By a final union bound, with probability at least 1/4 we have a cover with the weight at most $4 \ln m \cdot OPT$.

CONCLUSION

- Algorithm 1 returns a valid set cover with weight at most $4 \ln m$ times the optimal weight set-cover with probability of failure at most $1/4$. Thus, after running it at most 4 times in expectation, we'd expect to have a cover that has total weight at most $4 \ln m \cdot OPT$.
- It critically relies on a solution to the LP to guide the randomized part of the algorithm.
- Next, we'll see another example of this approach for the MAX-SAT problem

MAX-k-SAT

- Imagine that we have some CNF boolean function, where each clause has exactly k literals for some integer k.
- Each clause C_j has a set of positive variables P_j and a set of negative variables N_j
- Our goal is to set truth values to the variables in order to maximize the number of satisfied clauses
- IDEA: Solve an LP; Use the settings in this solution to assign probabilities to indicator r.v.'s; Round these r.v.'s.

The Linear Program (LP)

Maximize: $\sum_j z_j$

Subject to:

$z_j \leq \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i)$, $\forall C_j$

$0 \leq y_i \leq 1$, $\forall y_i$

$0 \leq z_j \leq 1$, $\forall z_j$

The Algorithm

- Write an LP for the boolean formula as in the previous slide
- Let y_i^* be the settings found in the solution found for the LP
- For each variable i, set i to TRUE with probability y_i^* and FALSE otherwise

Analysis Background

- Convex/Concave Functions
- Arithmetic/Geometric Mean inequality

Convex Functions

- A function, f, is **convex** if for all inputs x and y and for all $\lambda \in [0, 1]$:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

- Key fact: If f has a second derivative, then f is convex iff the second derivative is always non-negative.

Concave Functions

- A concave function is the negative of a convex function
- A function, f, is **concave** if for all inputs x and y and for all $\lambda \in [0, 1]$:

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$$

- Key fact: If f has a second derivative, then f is concave iff the second derivative is always negative.

GM $\leq$ AM

- For any non-negative $x_1, x_2, \ldots, x_k$, the geometric mean is at most equal to the arithmetic mean
- $(x_1 x_2 \ldots x_k)^{1/k} \leq (1/k)(x_1 + x_2 + \ldots + x_k)$
- Easy to see this for 2 variables: $\sqrt{xy} \leq (1/2)(x + y)$

Probability C_j is not satisfied

- Fix some clause C_j and let P_j be the set of positive and N_j be the set of negative variables in C_j
- Then the probability that the clause is not satisfied is

$$\begin{aligned}
\prod_{i \in P_j} (1 - y_i^*) \prod_{i \in N_j} y_i^* &\leq \left(\frac{1}{k} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right)^k \\
&= \left(1 - \frac{1}{k} \left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right) \right)^k \\
&\leq \left(1 - \frac{z_j^*}{k} \right)^k
\end{aligned}$$

First inequality holds since GM $\leq$ AM.

Using Concavity

- Probability that C_j *is* satisfied is: $1 - \left(1 - \frac{z_j^*}{k}\right)^k$
- $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{k}\right)^k$ is concave over $z_j^* \in [0, 1]$
- Hence: For any x and y and all $\lambda \in [0, 1]$:

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda) f(y)$$

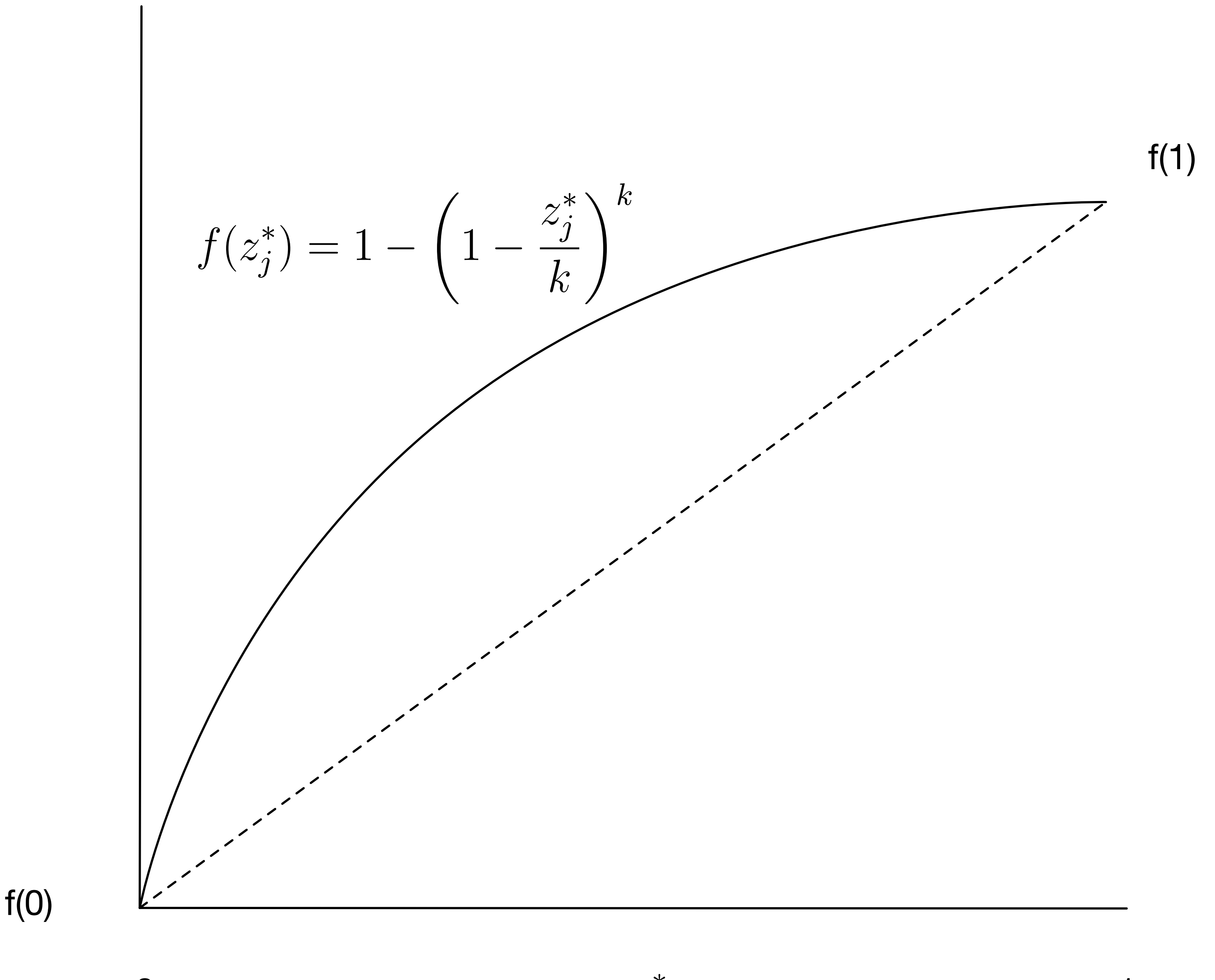
- Specifically if $x = 0$ and $y = 1$, then

$$f(1 - \lambda) \geq (1 - \lambda) f(1)$$

- Setting $1 - \lambda$ to be z_j^*, we get that

$$f(z_j^*) \geq z_j^* \left(1 - \left(1 - \frac{1}{k}\right)^k\right)$$

Bounding with Concave Property



Using Linearity of Expectation

- Probability that C_j is satisfied is $\geq z_j^* \left(1 - \left(1 - \frac{1}{k}\right)^k\right)$
- Let W be the number of clauses satisfied by our algorithm, and let W_j be an indicator r.v. that is 1 iff C_j is satisfied.

$$
\begin{aligned}
E(W) &= \sum_j E(W_j) \\
&\geq \sum_j z_j^* \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \\
&\geq \min_k \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \sum_j z_j^* \\
&\geq \min_k \left(1 - \left(1 - \frac{1}{k}\right)^k\right) OPT \\
&\geq (1 - 1/e) OPT \\
&\geq .632 \cdot OPT
\end{aligned}
$$

Step 5 of Last Slide

- For step 5 of last slide, note that since $1 - x \leq e^{-x}$:

$$(1 - 1/k)^k \leq e^{-1}$$

- So for any value of k,

$$1 - (1 - 1/k)^k \geq 1 - 1/e$$

- Just FYI, it's also true that

$$\lim_{k \to \infty} (1 - 1/k)^k = e^{-1}$$

Since

$$\lim_{k \to \infty} (1 + 1/k)^k = e$$

Take Away

- Many real-world problems can be shown to not have an efficient solution unless $P = NP$ (these are the NP-Hard problems)
- However, if a problem is shown to be NP-Hard, all hope is not lost!
- In many cases, we can come up with an provably good approximation algorithm for the NP-Hard problem.