CS 561, Dynamic Programming

Jared Saia
University of New Mexico

Dynamic Programming

- Intro
- String Alignment
- Matrix Multiplication
- Longest Common Subsequence

DP Intro

"Those who cannot remember the past are doomed to repeat it." - George Santayana, The Life of Reason, Book I: Introduction and Reason in Common Sense (1905)

What is Dynamic Programming?

- Dynamic Programming is basically "Divide and Conquer" with memorization
- Basic Trick is: *Don't solve the same problem more than once!*

Fibonacci Example

Consider the following procedure for computing the n-th Fibonacci number:

```
Fib(n){
  if (n<2)
    return 1;
  else
    return Fib(n-1) + Fib(n-2);
}
```

Analysis

- Q: What is the runtime of Fib?
- A: Except for recursive calls, the entire algorithm takes a constant number of steps. If $T(n)$ is the run time of the algorithm on input n, then we can say that:
  $T(0) = T(1) = 1$, $T(n) = T(n-2) + T(n-1) + 1$
- It's easy to show by induction that $T(n) = 2F_{n+1} - 1$. This is very bad!

Aside

- Q: How can we solve $T(n)$ exactly?
- A: We solved this recurrence using annihilators in the last lecture to get $T(n) = c_1\phi^n + c_2\widehat{\phi}^n + c_3 1^n$ where $\phi = \frac{1+\sqrt{5}}{2}$ and $\widehat{\phi} = \frac{1-\sqrt{5}}{2}$.

Aside II

- If we solve for constants, we get that:

$$\begin{aligned} T(0) = 1 \quad &= c_1 + c_2 + c_3 \\ T(1) = 1 \quad &= c_1\phi + c_2\widehat{\phi} + c_3 \\ T(2) = 2 \quad &= c_1\phi^2 + c_2\widehat{\phi}^2 + c_3 \end{aligned}$$

  Solving this system of linear equations (using Gaussian elimination) gives:

$$c_1 = 1 + \frac{1}{\sqrt{5}}, \quad c_2 = 1 - \frac{1}{\sqrt{5}}, \quad c_3 = -1,$$

Aside III

- So our final solution is

$$T(n) = \left(1 + \frac{1}{\sqrt{5}}\right)\phi^n + \left(1 - \frac{1}{\sqrt{5}}\right)\widehat{\phi}^n - 1 = \Theta(\phi^n).$$

The Problem

- The reason Fib is so slow is that it computes the same Fibonacci numbers over and over
- In general, there are F_{k-1} recursive calls to Fib(n-k)
- We can greatly speed up the algorithm by writing down the results of the recursive calls and looking them up if needed

DP-Fib

```
DP-Fib(n){
  if (n<2)
    return 1;
  else{
    if (F[n] is undefined){
      F[n] = DP-Fib(n-1) + DP-Fib(n-2);
    }
  return F[n];
}}
```

Analysis

- For every value of x between 1 and n, DP-Fib(x) is called exactly one time.
- Each call does constant work
- Thus runtime of DP-Fib(n) is $\Theta(n)$ - a *huge* savings

Take Away

Dynamic Programming is different than Divide and Conquer in the following way:

- "Divide and Conquer" divides problem into independent subproblems, solves the subproblems recursively and then combines solutions to solve original problem
- Dynamic Programming is used when the subproblems are not independent, i.e. the subproblems share subsubproblems
- For these kinds of problems, divide and conquer does more work than necessary
- Dynamic Programming solves each subproblem once only and saves the answer in a table for future reference

The Pattern

- **Formulate the problem recursively.**. Write down a formula for the whole problem as a simple combination of answers to smaller subproblems
- **Build solutions to your recurrence from the bottom up.** Write an algorithm that starts with the base cases of your recurrence and works its way up to the final solution by considering the intermediate subproblems in the correct order.

Note: Dynamic Programs store the results of intermediate subproblems. This is frequently *but not always* done with some type of table.

Edit Distance

- The *edit distance* between two words is the minimum number of letter insertions, letter deletions, and letter substitutions required to transform one word into another. For example, the edit distance between `FOOD` and `MONEY` is at most four:

$$\underline{\texttt{F}}\texttt{OOD} \rightarrow \texttt{MO}\underline{\texttt{O}}\texttt{D} \rightarrow \texttt{MON}\underline{\wedge}\texttt{D} \rightarrow \texttt{MONE}\underline{\texttt{D}} \rightarrow \texttt{MONEY}$$

String Alignment

Better way to display this process:

- Place the words one above the other in a table
- Put a gap in the first word for every insertion and a gap in the second word for every deletion
- Columns with two different characters correspond to substitutions
- Then the number of editing steps is just the number of columns that don't contain the same character twice

Example

- String Alignment for "FOOD" and "MONEY":

F	O	O		D
M	O	N	E	Y

- It's not too hard to see that we can't do better than four for the edit distance between "Food" and "Money"

Example II

- Unfortunately, it can be more difficult to compute the edit distance exactly. Example:

```
A  L  G  O  R     I     T  H  M
A  L     T  R  U  I  S  T  I  C
```

Key Observation

- If we remove the last column in an optimal alignment, the remaining alignment must also be optimal
- Easy to prove by contradiction: Assume there is some better subalignment of all but the last column. Then we can just paste the last column onto this better subalignment to get a better overall alignment.
- Note: The last column can be either: 1) a blank on top aligned with a character on bottom, 2) a character on top aligned with a blank on bottom or 3) a character on top aligned with a character on bottom

DP Solution

- To develop a DP algorithm for this problem, we first need to find a recursive definition
- Assume we have a m length string A and an n length string B
- Let $E(i, j)$ be the edit distance between the first i characters of A and the first j characters of B
- Then what we want to find is $E(n, m)$

Recursive Definition

- Say we want to compute $E(i, j)$ for some i and j
- Further say that the "Recursion Fairy" can tell us the solution to $E(i', j')$, for all $i' \leq i$, $j' \leq j$, *except* for $i' = i$ and $j' = j$
- Q: Can we compute $E(i, j)$ efficiently with help from our fairy friend?

Recursive Definition

There are three possible cases for the last column:

- **Blank on Bottom:** $E(i,j) = 1 + E(i-1,j)$
- **Blank on Top:** $E(i,j) = 1 + E(i,j-1)$
- **No Blank:** If $a_i = b_j$, $E(i,j) = E(i-1,j-1)$, else $E(i,j) = E(i-1,j-1) + 1$

Summary

Let $I(A[i] \neq B[j]) = 1$ if $A[i]$ and $B[j]$ are different, and 0 if they are the same. Then:

$$E(i,j) = \min \left\{ \begin{array}{l} E(i-1,j)+1, \\ E(i,j-1)+1, \\ E(i-1,j-1)+I(A[i] \neq B[j]) \end{array} \right\}$$

Base Case(s)

It's not too hard to see that:

- $E(0, j) = j$ for all j, since the j characters of B must be aligned with blanks
- Similarly, $E(i, 0) = i$ for all i

Recursive Alg

- We now have enough info to directly create a recursive algorithm
- The run time of this recursive algorithm would be given by the following recurrence:

$$T(m,0) = T(0,n) = O(1)$$

$$T(m,n) = T(m,n-1) + T(m-1,n) + T(n-1,m-1) + O(1)$$

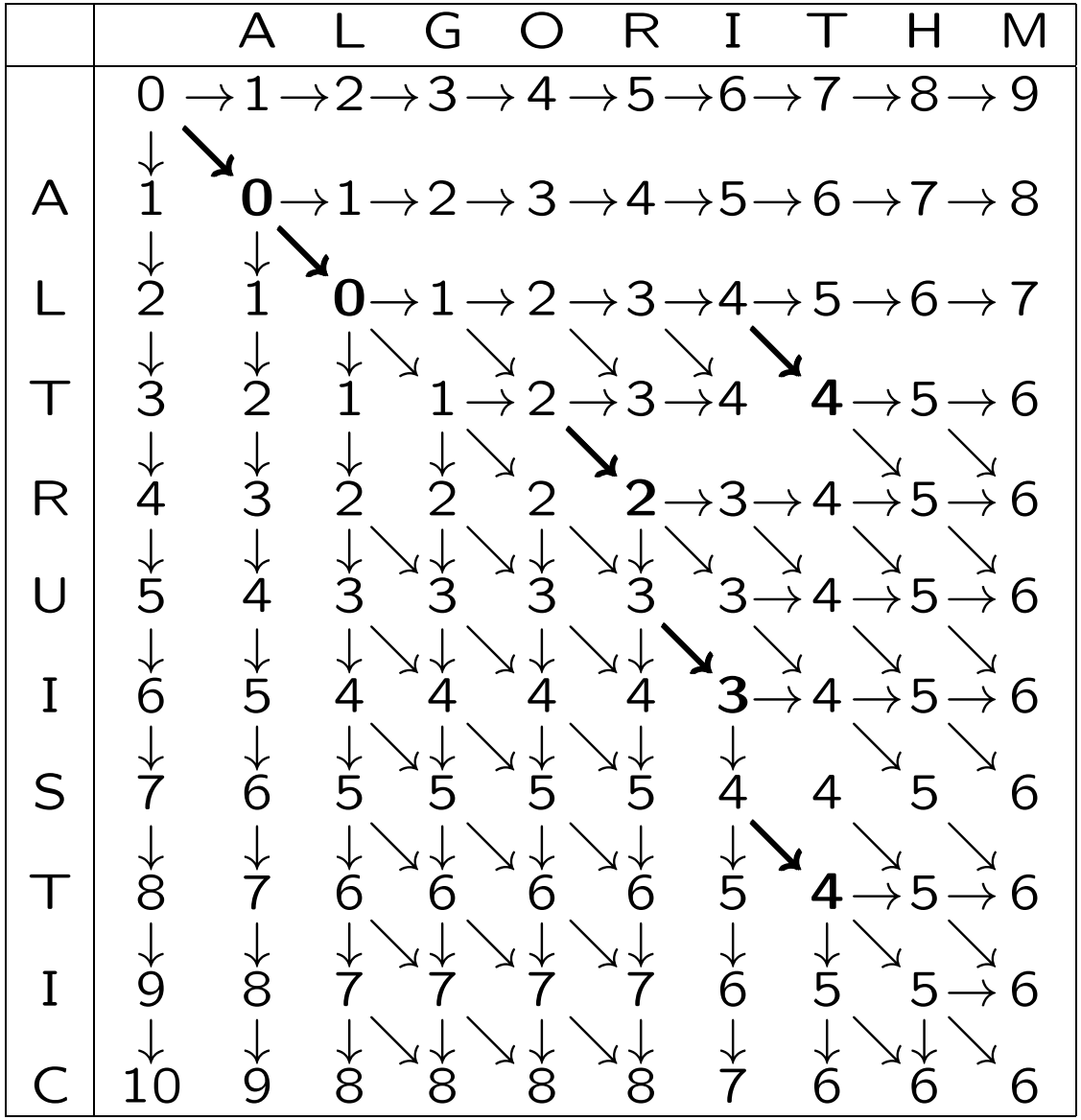
- Solution: $T(n,n) = \Theta(2^{n/2})$, which is terribly, terribly slow.

Better Idea

- We can build up a $m \times n$ table which contains all values of $E(i, j)$
- We start by filling in the base cases for this table: the entries in the 0-th row and 0-th column
- To fill in any other entry, we need to know the values directly above, to the left and above and to the left.
- Thus we can fill in the table in the standard way: left to right and top down to ensure that the entries we need to fill in each cell are always available

Example Table

- Bold numbers indicate places where characters in the strings are equal
- Arrows represent predecessors that define each entry: horizontal arrow is deletion, vertical is insertion and diagonal is substitution.
- Bold diagonal arrows are "free" substitutions of a letter for itself
- Any path of arrows from the top left to the bottom right corner gives an optimal alignment (there are three paths in this example table, so there are three optimal edit sequences).


A L G O R I T H M
0 1 2 3 4 5 6 7 8 9
A 1 0 1 2 3 4 5 6 7 8
L 2 1 0 1 2 3 4 5 6 7
T 3 2 1 1 2 3 4 4 5 6
R 4 3 2 2 2 2 3 4 5 6
U 5 4 3 3 3 3 3 4 5 6
I 6 5 4 4 4 4 3 4 5 6
S 7 6 5 5 5 5 4 4 5 6
T 8 7 6 6 6 6 5 4 5 6
I 9 8 7 7 7 7 6 5 5 6
C 10 9 8 8 8 8 7 6 6 6

The code

```
EditDistance(A[1,..,m],B[1,..,n]){
  for (i=1;i<=m;i++){
    Edit[i,0] = i;}
  for (j=1;j<=n;j++){
      Edit[0,j] = j;}
  for (i=1;i<=m;i++){
    for (j=1;j<=n;j++){
      if (A[i]==B[j]){
        Edit[i,j] = min(Edit[i-1,j]+1,
                        Edit[i,j-1]+1,
                        Edit[i-1,j-1]);
      }else{
        Edit[i,j] = min(Edit[i-1,j]+1,
                        Edit[i,j-1]+1;
                        Edit[i-1,j-1]+1);
  }}}
  return Edit[m,n];}
```

Analysis

- Let n be the length of the first string and m the length of the second string
- Then there are $\Theta(nm)$ entries in the table, and it takes $\Theta(1)$ time to fill each entry
- This implies that the run time of the algorithm is $\Theta(nm)$
- Q: Can you find a faster algorithm?

Reconstructing optimal alignment

- In this code, we do not keep info around to reconstruct an optimal alignment
- However, it is a simple matter to keep around another array which stores, for each cell, a pointer to the cell that was used to achieve the current cell's minimum edit distance
- To reconstruct a solution, we then need only follow these pointers from the bottom right corner up to the top left corner

In Class Exercise

- Create a string alignment table for the two strings "abba" and "bab". Put "abba" at the top of the table and "bab" on the left side
- Qi: ($i = 1, 2, \ldots, 5$) What is the i-th row of your table
- Q6: What is the minimum edit distance and how many alignments achieve it?

Take Away

- To solve the string alignment problem, we did the following: 1) formulated the problem recursively 2) built a solution to the recurrence from the bottom up
- Next we'll see how a similar technique can be used to solve the matrix multiplication problem.

Matrix Chain Multiplication

Problem:

- We are given a sequence of n matrices, $A_1, A_2, \ldots, A_n$, where for $i = 1, 2, \ldots, n$, matrix A_i has dimension p_{i-1} by p_i
- We want to compute the product, $A_1 A_2, \ldots, A_n$ as quickly as possible.
- In particular, we want to fully *parenthesize* the expression above so there are no ambiguities about how the matrices are multiplied
- A product of matrices is *fully parenthisized* if it is either a single matrix, or the product of two fully parenthesized matrix products, surrounded by parenthesis

Parenthesizing Matrices

- There are many ways to parenthesis the matrices
- Each way gives the same output (because of associativity of matrix multiplications)
- However the way we parenthesize will effect the *time* to compute the output
- Our Goal: Find a parenthesization which requires the minimal number of scalar multiplications

Example

- In this example, it's much better to multiply the last two matrices first (this gives us a short, narrow matrix on the right)
- Worse to multiply the first two matrices first (this gives us a short wide matrix on the left)
- In general, our goal is to find ways to always create narrow and short resulting matrices.

A Problem

Problem: There can be many ways to paranthesize. E.g.

- $(A_1(A_2(A_3A_4)))$
- $(A_1((A_2A_3)A_4))$
- $((A_1A_2)(A_3A_4))$
- $((A_1(A_2A_3))A_4)$
- $(((A_1A_2)A_3)A_4)$

A Problem

- Let $P(n)$ be the number of ways to paranthesize n matrices. Then $P(1) = 1$
- For $n \geq 2$, we know that a fully paranthesized product is the product of two fully paranthesized products, and the split can occur anywhere from $k = 1$ to $k = n - 1$.
- Hence for $n \geq 2$:

$$P(n) = \sum_{k=1}^{n-1} P(k)P(n-k)$$

- You can show that the solution to this recurrence is $\Omega(2^n)$

The Pattern

Q: Can we develop a DP Solution to this problem?

- **Formulate the problem recursively.**. Write down a formula for the whole problem as a simple combination of answers to smaller subproblems
- **Build solutions to your recurrence from the bottom up.** Write an algorithm that starts with the base cases of your recurrence and works its way up to the final solution by considering the intermediate subproblems in the correct order.

Key Observation

- Let $A_{i..j}$ (for $i \leq j$) be the matrix that results from evaluating the product $A_i A_{i+1}, \ldots A_j$
- Imagine we are computing $A_{i..j}$
- The last multiplication we do must look like this:

$$A_{i..j} = (A_{i..k}) * (A_{k+1..j})$$

  for some k between i and $j - 1$
- Then total cost to compute $A_{i..j}$ is:

  cost to compute $A_{i..k}$ +
  cost to compute $A_{k+1..j}$ +
  cost to multiply $A_{i..k}$ and $A_{k+1..j}$

Recursive Formulation

- For any integers x, y, let $m(x, y)$ be the minimum cost of computing $A_{x..y}$
- Then for any k between i and $j - 1$,

  $m(i, j) \leq$ optimal cost to compute $A_{i..k}$ +
  optimal cost to compute $A_{k+1..j}$ +
  cost to multiply $A_{i..k}$ and $A_{k+1..j}$
- In other words:

  $m(i, j) \leq m(i, k) +$
  $m(k + 1, j) +$
  cost to multiply $A_{i..k}$ and $A_{k+1..j}$

Cost to Multiply

- $A_{i..k}$ is a p_{i-1} by p_k matrix
- $A_{k+1..j}$ is a p_k by p_j matrix
- Thus multiplying $A_{i..k}$ and $A_{k+1..j}$ takes $p_{i-1}p_kp_j$ operations
- Hence we have:

$$\begin{aligned} m(i,j) \quad \leq \quad & m(i,k) + \\ & m(k+1,j) + \\ & p_{i-1}p_kp_j \end{aligned}$$

Recursive Formulation

- We've shown that $m(i,j) \leq m(i,k) + m(k+1,j) + p_{i-1}p_k p_j$ for any $k = i, i+1, \ldots, j-1$
- Further note that the optimal parenthesization must use some value of $k = i, i+1, \ldots, j-1$. So we need only pick the best
- Thus we have:

$$m(i,j) = 0 \text{ if } i = j$$
$$m(i,j) = \min_{i \leq k < j}\{m(i,k) + m(k+1,j) + p_{i-1}p_k p_j\}$$

The Recursive Algorithm

- We now have enough information to write a recursive function to solve the problem
- The recursive solution will have runtime given by the following recurrence:
- $T(1) = 1$,
- $T(n) = 1 + \sum_{k=1}^{n-1}(T(k) + T(n-k) + 1)$
- Unfortunately, the solution to this recurrence is $\Omega(2^n)$ (as shown on p. 346 of the text)

DP Algorithm

- Note that we must solve one subproblem for each choice of i and j satisfying $1 \leq i \leq j \leq n$
- This is only $\binom{n}{2} + n = \Theta(n^2)$ subproblems
- The recursive algorithm encounters each subproblem many times in the branches of the recursion tree.
- However, we can just compute these subproblems from the bottom up, storing the results in a table (this is the DP solution)

Pseudocode

```
Matrix-Chain-Order(int p[]){
  n = p.length - 1;
  for (i=1;i<=n;i++){
    m(i,i) = 0;
  }
  for (l=2;l<=n;l++){ \\l is chain length
    for (i=1;i<=n-l+1;i++){
      j = i+l-1;
      m[i,j] = MAXINT;
      for(k=i;k<=j-1;k++){
        q = m[i,k] + m[k+1,j] + p[i-1]*p[k]*p[j];
        if(q<m[i,j]){
          m[i,j] = q;
          s[i,j] = k;
        }
      }}}}
```

Pseudocode

- This code computes both the optimal cost and a parenthesization that achieves that cost
- It uses an m array to store the optimal costs of computing $m(i, j)$. It also uses a s array, where $s(i, j)$ stores the k value which gives $m(i, j)$
- The parenthesization can be recovered from the s array using the pseudocode in the book on p. 388.

Analysis

- This code has three nested loops, each of which takes on at most $n - 1$ values, and the inner loop takes $O(1)$ time.
- Thus the runtime is $O(n^3)$
- The algorithm also requires $\Theta(n^2)$ space

Example

- Consider the sequence of three matrices, A_1, A_2, A_3 whose dimensions are given by the sequence $3, 1, 2, 1$ (i.e. $p_0 = 3$, $p_1 = 1$, $p_2 = 2$, $p_3 = 1$)
- Let's construct the tables giving the optimal parenthesization
- The (i, j) entry of the first table will give the optimal cost for computing $A_{i..j}$, the (i, j) entry of the second table will give a k value which achieves this optimal cost

Computations

- $m(1,1) = m(2,2) = m(3,3) = 0$
- $m(1,2) = p_0 p_1 p_2 = 6$
- $m(2,3) = p_1 p_2 p_3 = 2$

Computations

$$
\begin{aligned}
m(1,3) &= \min \left\{ \begin{array}{l} m(1,1) + m(2,3) + p_0 p_1 p_3), \\ m(1,2) + m(3,3) + p_0 p_2 p_3) \end{array} \right\} \\
&= \min \left\{ \begin{array}{l} 0 + 2 + 3, \\ 6 + 0 + 6 \end{array} \right\} \\
&= 5
\end{aligned}
$$

Example, m array

	1	2	3
1	0	6	5
2	-	0	2
3	-	-	0

Example, s array

	1	2	3
1	-	1	1
2	-	-	2
3	-	-	-

Example

- Thus an optimal parenthesization is $(A_1(A_2A_3))$
- The cost of this is 5

Example II

- Consider the sequence of three matrices, A_1, A_2, A_3, A_4 whose dimensions are given by the sequence $3, 1, 2, 1, 2$ (i.e. $p_0 = 3$, $p_1 = 1$, $p_2 = 2$, $p_3 = 1$, $p_4 = 2$)
- Let's construct the tables giving the optimal parenthesization
- The (i, j) entry of the first table will give the optimal cost for computing $A_{i..j}$, the (i, j) entry of the second table will give a k value which achieves this optimal cost

Example II, m array

	1	2	3	4
1	0	6	5	10
2	-	0	2	4
3	-	-	0	4
4	-	-	-	0

Example II, s array

	1	2	3	4
1	-	1	1	1
2	-	-	2	3
3	-	-	-	3
4	-	-	-	-

Example Computation

$$
\begin{aligned}
m(1,4) &= \min \left\{ \begin{array}{l} m(1,1) + m(2,4) + p_0 p_1 p_4), \\ m(1,2) + m(3,4) + p_0 p_2 p_4), \\ m(1,3) + m(4,4) + p_0 p_3 p_4) \end{array} \right\} \\
&= \min \left\{ \begin{array}{l} 0 + 4 + 6, \\ 6 + 4 + 12, \\ 5 + 0 + 6 \end{array} \right\} \\
&= 10
\end{aligned}
$$

This minimum is achieved when $k = 1$

Example II

- Thus an optimal parenthesization is $(A_1((A_2A_3)A_4))$
- The cost of this is 10

In-Class Exercise

- Consider the sequence of three matrices, A_1, A_2, A_3 whose dimensions are given by the sequence $1, 2, 1, 2$ (i.e. $p_0 = 1$, $p_1 = 2$, $p_2 = 1$, $p_3 = 2$)
- Q1: What are the m array and s array for these inputs?
- Q2: What is the optimal parenthesization?

Subsequence Definition

- Assume given sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$
- Let $Z = \langle z_1, z_2, \ldots, z_l \rangle$
- Then Z is a *subsequence* of X if there exists a strictly increasing sequence $\langle i_1, i_2, \ldots, i_k \rangle$ of indices such that for all $j = 1, 2, \ldots, k$, $x_{i_j} = z_j$

Example

- Let $X = \langle A, B, C, B, A, B, D, C \rangle$,
- $Z = \langle A, C, A, B, C \rangle$
- Then, Z is a subsequence of X

Common Subsequence

- Given two sequences X and Y, we say that Z is a common subsequence of X and Y if Z is a subsequence of X and Z is a subsequence of Y
- Example: $X = \langle A, B, D, C, B, A, B, C \rangle$, $Y = \langle A, D, B, C, D, B, A, B \rangle$
- Then $Z = \langle A, B, B, A, B \rangle$ is a common subsequence
- Z is not a longest common subsequence(LCS) of X and Y though since the common subsequence $Z' = \langle A, B, C, B, A, B \rangle$ is longer
- Q: Is Z' a longest common subsequence?

LCS Problem

- We are given two sequences $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$
- Goal: Find a maximum-length common subsequence of X and Y

Brute Force

- Brute Force approach is to enumerate all possible subsequences of X, check to see if its a subsequence of Y, and then keep track of the longest common subsequence of both X and Y
- This is slow.
- Q: How many subsequences of X are there?

Terminology

- Given a sequence $X = \langle x_1, x_2, \ldots x_m \rangle$, for $i = 0, 1, \ldots, m$, let X_i be the i-th prefix of X i.e. $X_i = \langle x_1, x_2, \ldots, x_i \rangle$
- Example: if $X = \langle A, B, D, C \rangle$, $X_0 = \langle\rangle$ and $X_3 = \langle A, B, D \rangle$

Optimal Substructure

Lemma 1: Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and let $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y. Then:

- If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is a LCS of X_{m-1} and Y_{n-1}
- If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is a LCS of X_{m-1} and Y
- If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}

In-Class Exercise

- Prove each of the three statements in the previous slide
- Hint: Use proof by contradiction

Recursive Solution

- Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be arbitrary sequences
- Based on Lemma 1, there are two main possibilities for the LCS of X and Y:
  - If $x_m = y_n$, $LCS(X, Y)$ is $LCS(X_{m-1}, Y_{n-1})$ appended to $x_m = y_n$
  - Otherwise, either $LCS(X, Y)$ is $LCS(X_{m-1}, Y)$ or $LCS(X, Y_{n-1})$ (whichever is larger)

Recursive solution

- Let $c(i, j)$ be the length of an LCS of the sequence X_i, Y_j
- Note that $c(i, j) = 0$ if i or j is 0
- Thus we have:

$c(i, j) = 0$	if $i = 0$ or $j = 0$
$c(i, j) = c(i - 1, j - 1) + 1$	if $i, j > 0$ and $x_i = y_j$
$c(i, j) = \max(c(i, j - 1), c(i - 1, j))$	if $i, j > 0$ and $x_i \neq y_j$

DP Solution

- This is already enough to write up a recursive function, however the naive recursive function will take exponential time
- Instead, we can use dynamic programming and solve from the bottom up
- Code for doing this is on p. 353 and 355 of the text, basically it uses the same ideas we've seen before of filling in entries in a table from the bottom up.

Example

- Consider $X = \langle A, B, D, C, B, A, B, C \rangle$, $Y = \langle A, D, B, C, D, B, A, B \rangle$
- The next slide gives the table constructed by the DP algorithm for computing the LCS of X and Y
- The bold numbers represent one possible path giving a LCS.
- The arrows keep track of where the minimum is obtained from

Example

		A	B	D	C	B	A	B	C
	0	0	0	0	0	0	0	0	0
A	0	**1**	1	1	1	1	1	1	1
D	0	**1**	1	2	2	2	2	2	2
B	0	1	**2**	**2**	2	3	3	3	3
C	0	1	2	2	**3**	3	3	3	4
D	0	1	2	3	3	3	3	3	4
B	0	1	2	3	3	**4**	4	4	4
A	0	1	2	3	3	4	**5**	5	5
B	0	1	2	3	3	4	5	**6**	**6**

Reconstruction

- To find a reconstruction, first find a path of edges leading from the bottom right corner to the top left corner
- In this path, the target of each diagonal arrow gives a character to include in the LCS
- In our example, $\langle A, B, C, B, A, B \rangle$ is the LCS we get by following the edges along the only path from the bottom right to the top left

Take Away

- We've seen four different DP type algorithms
- In each case, we did the following 1) found a recurrence for the solution 2) built solutions to the recurrence from the bottom up
- You should be prepared to do this on your own now!