CS 561, Pre Lecture 1

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Today’s Outline

- Background
- Asymptotic Analysis
Why study algorithms?

“Seven years of College down the toilet” - John Belushi in Animal House

Can you get a job without knowing algorithms? Yes, but:

- You won’t understand why software systems work the way they do
- You won’t learn the fundamentals of systematic thinking (aka computation/problem solving)
- You’ll have less fun!
Why study algorithms? (II)

- Almost all big companies want programmers with knowledge of algorithms: Google, Facebook, Amazon, Oracle, Yahoo, Sandia, Los Alamos, etc.
- In most programming job interviews, they will ask you several questions about algorithms and/or data structures
- Your knowledge of algorithms will set you apart from the large masses of interviewees who know only how to program
- If you want to start your own company: many startups are successful because they’ve found better algorithms for solving a problem (e.g. Google, OpenAI, Akamai, etc.)
Why Study Algorithms? (III)

- You’ll improve your research skills in almost any area
- You’ll write better, faster code
- You’ll learn to think more abstractly and mathematically
- It’s one of the most challenging and interesting area of CS!
The following is a real job interview question (thanks to Maksim Noy):

- You are given an array with integers between 1 and 1,000,000.
- All integers between 1 and 1,000,000 are in the array at least once, and one of those integers is in the array twice.
- Q: Can you determine which integer is in the array twice? Can you do it while iterating through the array only once?
Solution

- Ideas on how to solve this problem?? What if we allowed multiple iterations?
Naive Algorithm

- Create a new array of ints between 1 and 1,000,000, which we’ll use to count the occurrences of each number. Initialize all entries to 0
- Go through the input array and each time a number is seen, update its count in the new array
- Go through the count array and see which number occurs twice.
- Return this number
Naive Algorithm Analysis

- Q: How long will this algorithm take?
- A: We iterate through the numbers 1 to 1,000,000 *three* times!
- Note that we also use up a lot of space with the extra array
- This is wasteful of time and space, particularly as the input array gets very large (e.g. it might be a huge data stream)
- Q: Can we do better?
Ideas for a better Algorithm

- Note that $\sum_{i=1}^{n} i = (n + 1)n/2$
- Let $S$ be the sum of the input array
- Let $x$ be the value of the repeated number
- Then $S = (1,000,000 + 1)\frac{1,000,000}{2} + x$
- Thus $x = S - (1,000,000 + 1)\frac{1,000,000}{2}$
A better Algorithm

- Iterate through the input array, summing up all the numbers, let $S$ be this sum
- Let $x = S - (1,000,000 + 1)1,000,000/2$
- Return $x$
Analysis

- This algorithm takes iterates through the input array just once
- It uses up essentially no extra space
- It is at least three times faster than the naive algorithm
- Further, if the input array is so large that it won’t fit in memory, this is the only algorithm which will work!
- These time and space bounds are the best possible
Take Away

- Designing good algorithms matters!
- Not always this easy to improve an algorithm
- However, with some thought and work, you can almost always get a better algorithm than the naive approach
How to analyze an algorithm?

- There are several resource bounds we could be concerned about: time, space, communication bandwidth, logic gates, etc.
- However, we are usually most concerned about time
- Recall that algorithms are independent of programming languages and machine types
- Q: So how do we measure resource bounds of algorithms
Random-access machine model

• We will use RAM model of computation in this class
• All instructions operate in serial
• All basic operations (e.g. add, multiply, compare, read, store, etc.) take unit time
• All “atomic” data (chars, ints, doubles, pointers, etc.) take unit space
Worst Case Analysis

- We’ll generally be pessimistic when we evaluate resource bounds
- We’ll evaluate the run time of the algorithm on the worst possible input sequence
- Amazingly, in most cases, we’ll still be able to get pretty good bounds
- Justification: The “average case” is often about as bad as the worst case.
Example Analysis

- Let’s consider the more general problem of the duplicate number problem, where the numbers are 1 to $n$ instead of 1 to 1,000,000.
Algorithm 1

- Create a new "count" array of ints of size $n$, which we’ll use to count the occurrences of each number. Initialize all entries to 0
- Go through the input array and each time a number is seen, update its count in the "count" array
- As soon as a number is seen in the input array which has already been counted once, return this number
Algorithm 2

- Iterate through the input array, summing up all the numbers, let $S$ be this sum
- Let $x = S - (n + 1)n/2$
- Return $x$
Example Analysis: Time

- Worst case: Algorithm 1 does $5 \times n$ operations ($n$ inits to 0 in “count” array, $n$ reads of input array, $n$ reads of “count” array (to see if value is 1), $n$ increments, and $n$ stores into count array)
- Worst case: Algorithm 2 does $2 \times n + 4$ operations ($n$ reads of input array, $n$ additions to value $S$, 4 computations to determine $x$ given $S$)
Example Analysis: Space

- Worst Case: Algorithm 1 uses $n$ additional units of space to store the “count” array
- Worst Case: Algorithm 2 uses 2 additional units of space
A Simpler Analysis

- Analysis above can be tedious for more complicated algorithms
- In many cases, we don’t care about constants. $5n$ is about the same as $2n + 4$ which is about the same as $an + b$ for any constants $a$ and $b$
- However we do still care about the difference in space: $n$ is very different from 2
- Asymptotic analysis is the solution to removing the tedium but ensuring good analysis
Asymptotic analysis?

- A tool for analyzing time and space usage of algorithms
- Assumes input size is a variable, say $n$, and gives time and space bounds as a function of $n$
- Ignores multiplicative and additive constants
- Concerned only with the rate of growth
- E.g. Treats run times of $n$, $10,000 \times n + 2000$, and $.5n + 2$ all the same (We use the term $O(n)$ to refer to all of them)
What is Asymptotic Analysis?(II)

• Informally, $O$ notation is the leading (i.e. quickest growing) term of a formula with the coefficient stripped off.
• $O$ is sort of a relaxed version of “≤”
• E.g. $n$ is $O(n)$ and $n$ is also $O(n^2)$
• By convention, we use the smallest possible $O$ value i.e. we say $n$ is $O(n)$ rather than $n$ is $O(n^2)$
More Examples

• E.g. $n$, $10,000n - 2000$, and $.5n + 2$ are all $O(n)$
• $n + \log n$, $n - \sqrt{n}$ are $O(n)$
• $n^2 + n + \log n$, $10n^2 + n - \sqrt{n}$ are $O(n^2)$
• $n \log n + 10n$ is $O(n \log n)$
• $10 \times \log^2 n$ is $O(\log^2 n)$
• $n\sqrt{n} + n \log n + 10n$ is $O(n\sqrt{n})$
• $10,000$, $2^{50}$ and $4$ are $O(1)$
More Examples

- Algorithm 1 and 2 both take time $O(n)$
- Algorithm 1 uses $O(n)$ extra space
- But, Algorithm 2 uses $O(1)$ extra space
Formal Defn of Big-O

• A function $f(n)$ is $O(g(n))$ if there exist positive constants $c$ and $n_0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$
Example

- Let’s show that $f(n) = 10n + 100$ is $O(g(n))$ where $g(n) = n$
- We need to give constants $c$ and $n_0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$
- In other words, we need constants $c$ and $n_0$ such that $10n + 100 \leq cn$ for all $n \geq n_0$
Example

• We can solve for appropriate constants:

\[ 10n + 100 \leq cn \quad (1) \]
\[ 10 + \frac{100}{n} \leq c \quad (2) \]

• So if \( n > 1 \), then \( c \) should be greater than 110.
• In other words, for all \( n > 1 \), \( 10n + 100 \leq 110n \)
• So \( 10n + 100 \) is \( O(n) \)
Questions

Express the following in $O$ notation

- $n^3/1000 - 100n^2 - 100n + 3$
- $\log n + 100$
- $10 \times \log^2 n + 100$
- $\sum_{i=1}^{n} i$
Relatives of big-O

The following are relatives of big-O:

- $O$ “≤”
- $\Theta$ “≈”
- $\Omega$ “≥”
- $o$ “≪”
- $\omega$ “≪”
- $\exists$
Formal Defns

• $O(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$

• $\Theta(g(n)) = \{f(n) : \text{there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \text{ for all } n \geq n_0\}$

• $\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0\}$
Formal Defns (II)

- $o(g(n)) = \{ f(n) : \text{for any positive constant } c > 0 \text{ there exists } n_0 > 0 \text{ such that } 0 \leq f(n) < cg(n) \text{ for all } n \geq n_0 \}$

- $\omega(g(n)) = \{ f(n) : \text{for any positive constant } c > 0 \text{ there exists } n_0 > 0 \text{ such that } 0 \leq cg(n) < f(n) \text{ for all } n \geq n_0 \}$
Relatives of big-O

When would you use each of these? Examples:

- $O$ “≤” This algorithm is $O(n^2)$ (i.e. worst case is $\Theta(n^2)$)
- $\Theta$ “=” This algorithm is $\Theta(n)$ (best and worst case are $\Theta(n)$)
- $\Omega$ “≥” Any comparison-based algorithm for sorting is $\Omega(n \log n)$
- $o$ “<” Can you write an algorithm for sorting that is $o(n^2)$?
- $\omega$ “>” This algorithm is not linear, it can take time $\omega(n)$
Rule of Thumb

• Let \( f(n) \), \( g(n) \) be two functions of \( n \)
• Let \( f_1(n) \), be the fastest growing term of \( f(n) \), stripped of its coefficient.
• Let \( g_1(n) \), be the fastest growing term of \( g(n) \), stripped of its coefficient.

Then we can say:

• If \( f_1(n) \leq g_1(n) \) then \( f(n) = O(g(n)) \)
• If \( f_1(n) \geq g_1(n) \) then \( f(n) = \Omega(g(n)) \)
• If \( f_1(n) = g_1(n) \) then \( f(n) = \Theta(g(n)) \)
• If \( f_1(n) < g_1(n) \) then \( f(n) = o(g(n)) \)
• If \( f_1(n) > g_1(n) \) then \( f(n) = \omega(g(n)) \)
More Examples

The following are all true statements:

• $\sum_{i=1}^{n} i^2$ is $O(n^3)$, $\Omega(n^3)$ and $\Theta(n^3)$
• $\log n$ is $o(\sqrt{n})$
• $\log n$ is $o(\log^2 n)$
• $10,000n^2 + 25n$ is $\Theta(n^2)$
Problems

True or False? (Justify your answer)

• $n^3 + 4$ is $\omega(n^2)$
• $n \log n^3$ is $\Theta(n \log n)$
• $\log^3 5n^2$ is $\Theta(\log n)$
• $10^{-10}n^2 + n$ is $\Theta(n)$
• $n \log n$ is $\Omega(n)$
• $n^3 + 4$ is $o(n^4)$
Another Example

• Let \( f(n) = 10 \log^2 n + \log n \), \( g(n) = \log^2 n \). Let’s show that \( f(n) = \Theta(g(n)) \).

• We want positive constants \( c_1, c_2 \) and \( n_0 \) such that \( 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \) for all \( n \geq n_0 \)

\[
0 \leq c_1 \log^2 n \leq 10 \log^2 n + \log n \leq c_2 \log^2 n
\]

Dividing by \( \log^2 n \), we get:

\[
0 \leq c_1 \leq 10 + 1/\log n \leq c_2
\]

• If we choose \( c_1 = 1 \), \( c_2 = 11 \) and \( n_0 = 2 \), then the above inequality will hold for all \( n \geq n_0 \)
Show that for $f(n) = n + 100$ and $g(n) = (1/2)n^2$, that $f(n) \neq \Theta(g(n))$

- What statement would be true if $f(n) = \Theta(g(n))$?
- Show that this statement can not be true.