

CS 561: Homework 5

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Remember: You are encouraged to work on the homework in groups, but please observe the “Star Trek” rule from the syllabus.

1. A frog jumps along an infinite line of lily pads numbered $1, 2, 3, \dots$. It starts on lily pad 1. Whenever it is on lily pad i , it jumps to lily pad $i + 1$ with probability $1/2$ and to lily pad $i + 2$ with probability $1/2$.
 - (a) For $i \geq 1$, let $p(i)$ be the probability that the frog ever visits lily pad i . Give a recurrence for $p(i)$, including all necessary base cases.
 - (b) Use the annihilator method to find the general form of the solution to the recurrence.
 - (c) Use the base cases to obtain an exact formula for $p(i)$, and check that your formula satisfies both the recurrence and the base cases.
 - (d) For a fixed integer $n \geq 1$, let X be the number of lily pads among $1, 2, \dots, n$ that the frog visits. Using indicator random variables, linearity of expectation, and your formula for $p(i)$, compute $E[X]$ exactly.
2. **Creaky Staircase.** A staircase has steps $1, 2, \dots, n$. Landing on step i incurs a nonnegative cost c_i . You start at position 0, just below the staircase, and the top is position $n + 1$, which has cost 0. From any position, you may move upward by exactly 2 or 3 positions, but you may not move past position $n + 1$. Your goal is to reach the top while minimizing the total cost of the steps on which you land.
 - (a) Consider the greedy strategy that, at each position, chooses the lower-cost destination among the currently legal moves (breaking ties by choosing the smaller-numbered destination). Give an example in which this strategy reaches the top but is not optimal.
 - (b) Define, in words, a function whose values for smaller subproblems can be used to solve the problem. Then give a recurrence for this function, including base cases for unreachable positions.
 - (c) Describe a dynamic program based on your recurrence. State the order in which the table is filled, the value returned, and the running time and space usage.

3. A thief is planning robberies over the next n days. Robbing the bank on day i yields revenue $b_i \geq 0$ and requires an integer amount of work $w_i \geq 1$. Any two robbery days must differ by at least 10, and the total work used may not exceed an integer budget $W \geq 0$.

For $0 \leq i \leq n$ and $0 \leq j \leq W$, let $r(i, j)$ be the maximum total revenue obtainable using only days $1, 2, \dots, i$ and at most j units of work.

- (a) Give a recurrence for $r(i, j)$, including all necessary base cases. Your recurrence should explicitly handle the case $w_i > j$.
 - (b) Briefly justify your recurrence by considering whether an optimal solution robs the bank on day i .
 - (c) Describe a dynamic program based on the recurrence. State the table dimensions, the order in which it is filled, the value returned, and the running time and space usage.
4. **Chocolate with Friends.** A chocolate bar consists of n chunks arranged in a row. Chunk i has positive value v_i . You must divide the bar into exactly k nonempty contiguous parts, where $1 \leq k \leq n$. After seeing the division, your friends take the $k - 1$ most valuable parts, leaving you a least-valuable part. Thus, you want to maximize the minimum value of any part.

Let $S(i) = \sum_{q=1}^i v_q$, with $S(0) = 0$. For $1 \leq j \leq i \leq n$, define $F(i, j)$ to be the maximum possible value of the least-valuable part when the first i chunks are divided into exactly j nonempty contiguous parts.

- (a) Give a recurrence for $F(i, j)$, including the base case. Your recurrence should consider the location of the final cut.
- (b) Justify why your recurrence considers every feasible division and returns the optimal value.
- (c) Describe a dynamic program based on the recurrence. State the table dimensions, the order in which it is filled, the value returned, and the running time and space usage. Assume that all prefix sums $S(i)$ have been precomputed.