CS 561, Lecture 11

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Skip List

- Technically, not a BST, but they implement all of the same operations
- Very elegant randomized data structure, simple to code but analysis is subtle
- They guarantee that, with high probability, all the major operations take $O(\log n)$ time

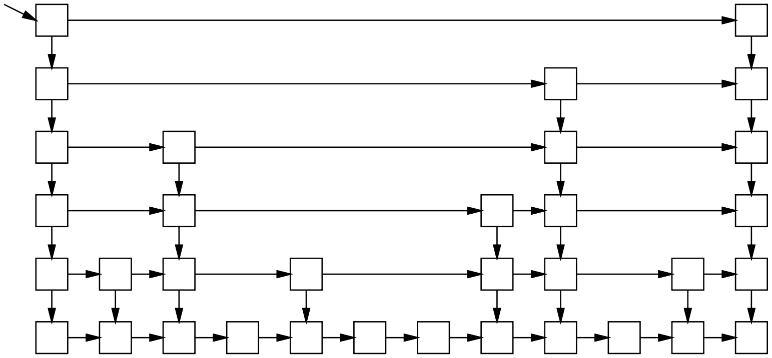
Skip List

- A skip list is basically a collection of doubly-linked lists, $L_1, L_2, \ldots, L_x$, for some integer x
- Each list has a special head and tail node, the keys of these nodes are assumed to be $-$MAXNUM and $+$MAXNUM respectively
- The keys in each list are in sorted order (non-decreasing)

Skip List

- Every node is stored in the bottom list
- For each node in the bottom list, we flip a coin over and over until we get tails. For each heads, we make a duplicate of the node.
- The duplicates are stacked up in levels and the nodes on each level are strung together in sorted linked lists
- Each node v stores a search key (key(v)), a pointer to its next lower copy (down(v)), and a pointer to the next node in its level (right(v)).

Example



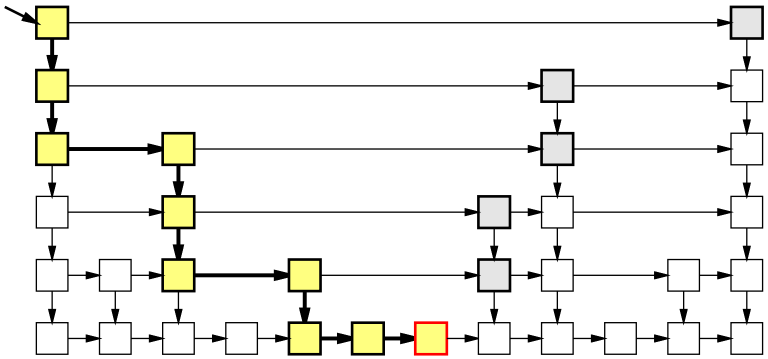
Search

- To do a search for a key, x, we start at the leftmost node L in the highest level
- We then scan through each level as far as we can without passing the target value x and then proceed down to the next level
- The search ends either when we find the key x or fail to find x on the lowest level

Search

```
SkipListFind(x, L){
  v = L;
  while (v != NULL) and (Key(v) != x){
    if (Key(Right(v)) > x)
      v = Down(v);
    else
      v = Right(v);
  }
return v;
}
```


Search Example



Insert

p is a constant between 0 and 1, typically $p = 1/2$, let rand() return a random value between 0 and 1

```
Insert(k){
First call Search(k), let pLeft be the leftmost elem <= k in L_1
Insert k in L_1, to the right of pLeft
i = 2;
while (rand()<= p){
  insert k in the appropriate place in L_i;
}
```


Deletion

- Deletion is very simple
- First do a search for the key to be deleted
- Then delete that key from all the lists it appears in from the bottom up, making sure to "zip up" the lists after the deletion

Analysis

- Intuitively, each level of the skip list has about half the number of nodes of the previous level, so we expect the total number of levels to be about $O(\log n)$
- Similarly, each time we add another level, we cut the search time in half except for a constant overhead
- So after $O(\log n)$ levels, we would expect a search time of $O(\log n)$
- We will now formalize these two intuitive observations

Height of Skip List

- For some key, i, let X_i be the maximum height of i in the skip list.
- Q: What is the probability that $X_i \geq 2\log n$?
- A: If $p = 1/2$, we have:

$$\begin{aligned} P(X_i \geq 2\log n) &= \left(\frac{1}{2}\right)^{2\log n} \\ &= \frac{1}{(2^{\log n})^2} \\ &= \frac{1}{n^2} \end{aligned}$$

- Thus the probability that a particular key i achieves height $2\log n$ is $\frac{1}{n^2}$

Height of Skip List

- Q: What is the probability that *any* key achieves height $2\log n$?
- A: We want

$$P(X_1 \geq 2\log n \text{ or } X_2 \geq 2\log n \text{ or } \ldots \text{ or } X_n \geq 2\log n)$$

- By a Union Bound, this probability is no more than

$$P(X_1 \geq k\log n) + P(X_2 \geq k\log n) + \cdots + P(X_n \geq k\log n)$$

- Which equals:

$$\sum_{i=1}^{n} \frac{1}{n^2} = \frac{n}{n^2} = 1/n$$

Height of Skip List

- This probability gets small as n gets large
- In particular, the probability of having a skip list of size exceeding $2\log n$ is $o(1)$
- If an event occurs with probability $1 - o(1)$, we say that it occurs *with high probability*
- *Key Point:* The height of a skip list is $O(\log n)$ with high probability.

In-Class Exercise Trick

A trick for computing expectations of discrete positive random variables:

- Let X be a discrete r.v., that takes on values from 1 to n

$$E(X) = \sum_{i=1}^{n} P(X \geq i)$$

Why?

$$\begin{aligned}
\sum_{i=1}^{n} P(X \geq i) &= P(X=1) + P(X=2) + P(X=3) + \ldots \\
&+ P(X=2) + P(X=3) + P(X=4) + \ldots \\
&+ P(X=3) + P(X=4) + P(X=5) + \ldots \\
&+ \ldots \\
&= 1 * P(X=1) + 2 * P(X=2) + 3 * P(X=3) + \ldots \\
&= E(X)
\end{aligned}$$

In-Class Exercise

Q: How much memory do we expect a skip list to use up?

- Let X_i be the number of lists that element i is inserted in.
- Q: What is $P(X_i \geq 1)$, $P(X_i \geq 2)$, $P(X_i \geq 3)$?
- Q: What is $P(X_i \geq k)$ for general k?
- Q: What is $E(X_i)$?
- Q: Let $X = \sum_{i=1}^{n} X_i$. What is $E(X)$?

Search Time

- Its easier to analyze the search time if we imagine running the search backwards
- Imagine that we start at the found node v in the bottommost list and we trace the path backwards to the top leftmost senitel, L
- This will give us the length of the search path from L to v which is the time required to do the search

Backwards Search

```
SLFback(v){
  while (v != L){
    if (Up(v)!=NIL)
      v = Up(v);
    else
      v = Left(v);
}}
```


Backward Search

- For every node v in the skip list Up(v) exists with probability 1/2. So for purposes of analysis, SLFBack is the same as the following algorithm:

```
FlipWalk(v){
  while (v != L){
    if (COINFLIP == HEADS)
      v = Up(v);
    else
      v = Left(v);
}}
```

Analysis

- For this algorithm, the expected number of heads is exactly the same as the expected number of tails
- Thus the expected run time of the algorithm is twice the expected number of upward jumps
- Since we already know that the number of upward jumps is $O(\log n)$ with high probability, we can conclude that the expected search time is $O(\log n)$