CS 561, Lecture 24

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Outline

- All Pairs Shortest Paths
- TSP Approximation Algorithm

All-Pairs Shortest Paths

- For the single-source shortest paths problem, we wanted to find the shortest path from a source vertex s to all the other vertices in the graph
- We will now generalize this problem further to that of finding the shortest path from *every* possible source to *every* possible destination
- In particular, for every pair of vertices u and v, we need to compute the following information:
  - $dist(u, v)$ is the length of the shortest path (if any) from u to v
  - $pred(u, v)$ is the second-to-last vertex (if any) on the shortest path (if any) from u to v

Example

- For any vertex v, we have $dist(v, v) = 0$ and $pred(v, v) = NULL$
- If the shortest path from u to v is only one edge long, then $dist(u, v) = w(u \to v)$ and $pred(u, v) = u$
- If there's no shortest path from u to v, then $dist(u, v) = \infty$ and $pred(u, v) = NULL$

APSP

- The output of our shortest path algorithm will be a pair of $|V| \times |V|$ arrays encoding all $|V|^2$ distances and predecessors.
- Many maps contain such a distance matric - to find the distance from (say) Albuquerque to (say) Ruidoso, you look in the row labeled "Albuquerque" and the column labeled "Ruidoso"
- In this class, we'll focus only on computing the distance array
- The predecessor array, from which you would compute the actual shortest paths, can be computed with only minor additions to the algorithms presented here

Lots of Single Sources

- Most obvious solution to APSP is to just run SSSP algorithm $|V|$ times, once for every possible source vertex
- Specifically, to fill in the subarray $dist(s, *)$, we invoke either Dijkstra's or Bellman-Ford starting at the source vertex s
- We'll call this algorithm ObviousAPSP

ObviousAPSP

```
ObviousAPSP(V,E,w){
  for every vertex s{
    dist(s,*) = SSSP(V,E,w,s);
  }
}
```


Analysis

- The running time of this algorithm depends on which SSSP algorithm we use
- If we use Bellman-Ford, the overall running time is $O(|V|^2|E|) = O(|V|^4)$
- If all the edge weights are positive, we can use Dijkstra's instead, which decreases the run time to $\Theta(|V||E|+|V|^2 \log |V|) = O(|V|^3)$

Problem

- We'd like to have an algorithm which takes $O(|V|^3)$ but which can also handle negative edge weights
- We'll see that a dynamic programming algorithm, the Floyd Warshall algorithm, will achieve this
- Note: the book discusses another algorithm, Johnson's algorithm, which is asymptotically better than Floyd Warshall on sparse graphs. However we will not be discussing this algorithm in class.

Dynamic Programming

- Recall: Dynamic Programming = Recursion + Memorization
- Thus we first need to come up with a recursive formulation of the problem
- We might recursively define $dist(u,v)$ as follows:

$$dist(u,v) = \begin{cases} 0 & \text{if } u = v \\ \min_x \big(dist(u,x) + w(x \to v)\big) & \text{otherwise} \end{cases}$$

The problem

- In other words, to find the shortest path from u to v, try all possible predecessors x, compute the shortest path from u to x and then add the last edge $u \to v$
- **Unfortunately, this recurrence doesn't work**
- To compute $dist(u,v)$, we first must compute $dist(u,x)$ for every other vertex x, but to compute any $dist(u,x)$, we first need to compute $dist(u,v)$
- We're stuck in an infinite loop!

The solution

- To avoid this circular dependency, we need some additional parameter that decreases at each recursion and eventually reaches zero at the base case
- One possibility is to include the number of edges in the shortest path as this third magic parameter
- So define $dist(u,v,k)$ to be the length of the shortest path from u to v that uses *at most* k edges
- Since we know that the shortest path between any two vertices uses at most $|V|-1$ edges, what we want to compute is $dist(u,v,|V|-1)$

The Recurrence

$$dist(u,v,k) = \begin{cases} 0 & \text{if } u = v \\ \infty & \text{if } k = 0 \text{ and } u \neq v \\ \min_x \big(dist(u,x,k-1) + w(x \to v)\big) & \text{otherwise} \end{cases}$$

The Algorithm

- It's not hard to turn this recurrence into a dynamic programming algorithm
- Even before we write down the algorithm, though, we can tell that its running time will be $\Theta(|V|^4)$
- This is just because the recurrence has four variables — u, v, k and x — each of which can take on $|V|$ different values
- Except for the base cases, the algorithm will just be four nested "for" loops

DP-APSP

```
DP-APSP(V,E,w){
  for all vertices u in V{
    for all vertices v in V{
      if(u=v)
         dist(u,v,0) = 0;
      else
         dist(u,v,0) = infinity;
  }}
  for k=1 to |V|-1{
    for all vertices u in V{
      for all vertices u in V{
        dist(u,v,k) = infinity;
          for all vertices x in V{
            if (dist(u,v,k)>dist(u,x,k-1)+w(x,v))
              dist(u,v,k) = dist(u,x,k-1)+w(x,v);
}}}}}
```


The Problem

- This algorithm still takes $O(|V|^4)$ which is no better than the ObviousAPSP algorithm
- If we use a certain divide and conquer technique, there is a way to get this down to $O(|V|^3 \log |V|)$ (think about how you might do this)
- However, to get down to $O(|V|^3)$ run time, we need to use a different third parameter in the recurrence

Floyd-Warshall

- Number the vertices arbitrarily from 1 to $|V|$
- Define $dist(u, v, r)$ to be the shortest path from u to v where all *intermediate* vertices (if any) are numbered r or less
- If $r = 0$, we can't use any intermediate vertices so shortest path from u to v is just the weight of the edge (if any) between u and v
- If $r > 0$, then either the shortest legal path from u to v goes through vertex r or it doesn't
- We need to compute the shortest path distance from u to v with no restrictions, which is just $dist(u, v, |V|)$

The recurrence

We get the following recurrence:

$$dist(u,v,r) = \begin{cases} w(u \to v) & \text{if } r = 0 \\ \min\{dist(u,v,r-1), \\ \qquad dist(u,r,r-1) + dist(r,v,r-1)\} & \text{otherwise} \end{cases}$$

The Algorithm

```
FloydWarshall(V,E,w){
  for u=1 to |V|{
    for v=1 to |V|{
      dist(u,v,0) = w(u,v);
  }}
  for r=1 to |V|{
    for u=1 to |V|{
      for v=1 to |V|{
        if (dist(u,v,r-1) < dist(u,r,r-1) + dist(r,v,r-1))
          dist(u,v,r) = dist(u,v,r-1);
        else
          dist(u,v,r) = dist(u,r,r-1) + dist(r,v,r-1);
}}}}
```


Analysis

- There are three variables here, each of which takes on $|V|$ possible values
- Thus the run time is $\Theta(|V|^3)$
- Space required is also $\Theta(|V|^3)$

Take Away

- Floyd-Warshall solves the APSP problem in $\Theta(|V|^3)$ time even with negative edge weights
- Floyd-Warshall uses dynamic programming to compute APSP
- We've seen that sometimes for a dynamic program, we need to introduce an *extra variable* to break dependencies in the recurrence.
- We've also seen that the choice of this extra variable can have a big impact on the run time of the dynamic program

TSP

- A version of the TSP problem is: "Given a weighted graph G, what is the shortest Hamiltonian Cycle of G?"
- Where a Hamiltonian Cycle is a path that visits each node in G exactly once and returns to the starting node
- This TSP problem is NP-Hard by a reduction from Hamiltonian Cycle
- However, there is a 2-approximation algorithm for this problem if the edge weights obey the *triangle inequality*

Triangle Inequality

- In many practical problems, it's reasonable to make the assumption that the weights, c, of the edges obey the *triangle inequality*
- The triangle inequality says that for all vertices $u, v, w \in V$:

$$c(u,w) \leq c(u,v) + c(v,w)$$

- In other words, the cheapest way to get from u to w is always to just take the edge (u,w)
- In the real world, this is often a pretty natural assumption. For example it holds if the vertices are points in a plane and the cost of traveling between two vertices is just the euclidean distance between them.

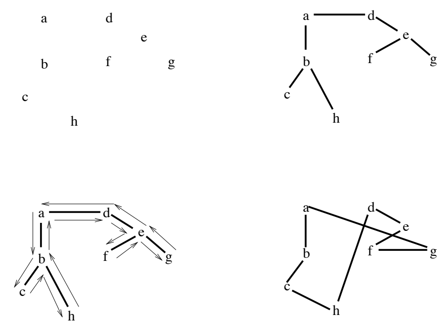
Approximation Algorithm

- Given a weighted graph G, the algorithm first computes a MST for G, T, and then arbitrarily selects a root node r of T.
- It then lets L be the list of the vertices visited in a depth first traversal of T starting at r.
- Finally, it returns the Hamiltonian Cycle, H, that visits the vertices in the order L.

Approximation Algorithm

```
Approx-TSP(G){
  T = MST(G);
  L = the list of vertices visited in a depth first traversal
      of T, starting at some arbitrary node in T;
  H = the Hamiltonian Cycle that visits the vertices in the
      order L;
  return H;
}
```


Example Run



The top left figure shows the graph G (edge weights are just the Euclidean distances between vertices); the top right figure shows the MST T. The bottom left figure shows the depth first walk on T, $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$; the bottom right figure shows the Hamiltonian cycle H obtained by deleting repeat visits from W, $H = (a, b, c, h, d, e, f, g)$.

Analysis

- The first step of the algorithm takes $O(|E| + |V| \log |V|)$ (if we use Prim's algorithm)
- The second step is $O(|V|)$
- The third step is $O(|V|)$.
- Hence the run time of the entire algorithm is polynomial

Analysis

An important fact about this algorithm is that: *the cost of the MST is less than the cost of the shortest Hamiltonian cycle.*

- To see this, let T be the MST and let $H*$ be the shortest Hamiltonian cycle.
- Note that if we remove one edge from $H*$, we have a spanning tree, T'
- Finally, note that $w(H*) \geq w(T') \geq w(T)$
- Hence $w(H*) \geq w(T)$

Analysis

- Now let W be a depth first walk of T which traverses each edge exactly twice (similar to what you did in the hw)
- In our example, $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$
- Note that $c(W) = 2c(T)$
- This implies that $c(W) \leq 2c(H*)$

Analysis

- Unfortunately, W is not a Hamiltonian cycle since it visits some vertices more than once
- However, we can delete a visit to any vertex and the cost will not increase *because of the triangle inequality*. (The path without an intermediate vertex can only be shorter)
- By repeatedly applying this operation, we can remove from W all but the first visit to each vertex, without increasing the cost of W.
- In our example, this will give us the ordering $H = (a, b, c, h, d, e, f, g)$

Analysis

- By the last slide, $c(H) \leq c(W)$.
- So $c(H) \leq c(W) = 2c(T) \leq 2c(H*)$
- Thus, $c(H) \leq 2c(H*)$
- In other words, the Hamiltonian cycle found by the algorithm has cost no more than twice the shortest Hamiltonian cycle.

Take Away

- Many real-world problems can be shown to not have an efficient solution unless $P = NP$ (these are the NP-Hard problems)
- However, if a problem is shown to be NP-Hard, all hope is not lost!
- In many cases, we can come up with an provably good approximation algorithm for the NP-Hard problem.