CS 561, Lecture 4

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Today's Outline

- Annihilators with Multiple Operators
- Annihilators for recurrences with non-homogeneous terms
- Transformations

Multiple Operators

- We can apply multiple operators to a sequence
- For example, we can multiply by the constant c and then by the constant d to get the operator cd
- We can also multiply by c and then shift left to get $c\mathbf{L}T$ which is the same as $\mathbf{L}cT$
- We can also shift the sequence twice to the left to get $\mathbf{LL}T$ which we'll write in shorthand as $\mathbf{L}^2T$

Multiple Operators

- We can string operators together to annihilate more complicated sequences
- Consider: $T = \langle 2^0 + 3^0, 2^1 + 3^1, 2^2 + 3^2, \cdots \rangle$
- We know that $(\mathbf{L}-2)$ annihilates the powers of 2 while leaving the powers of 3 essentially untouched
- Similarly, $(\mathbf{L} - 3)$ annihilates the powers of 3 while leaving the powers of 2 essentially untouched
- Thus if we apply both operators, we'll see that $(\mathbf{L}-2)(\mathbf{L}-3)$ annihilates the sequence T

The Details

- Consider: $T = \langle a^0 + b^0, a^1 + b^1, a^2 + b^2, \cdots \rangle$
- $\mathbf{L}T = \langle a^1 + b^1, a^2 + b^2, a^3 + b^3, \cdots \rangle$
- $aT = \langle a^1 + a * b^0, a^2 + a * b^1, a^3 + a * b^2, \cdots \rangle$
- $\mathbf{L}T - aT = \langle (b-a)b^0, (b-a)b^1, (b-a)b^2, \cdots \rangle$
- We know that $(\mathbf{L} - a)T$ annihilates the a terms and multiplies the b terms by $b - a$ (a constant)
- Thus $(\mathbf{L} - a)T = \langle (b-a)b^0, (b-a)b^1, (b-a)b^2, \cdots \rangle$
- And so the sequence $(\mathbf{L} - a)T$ is annihilated by $(\mathbf{L} - b)$
- Thus the annihilator of T is $(\mathbf{L} - b)(\mathbf{L} - a)$

Key Point

- In general, the annihilator $(\mathbf{L} - a)(\mathbf{L} - b)$ (where $a \neq b$) will anihilate *only* all sequences of the form $\langle c_1 a^n + c_2 b^n \rangle$
- We will often multiply out $(\mathbf{L} - a)(\mathbf{L} - b)$ to $\mathbf{L}^2 - (a + b)\mathbf{L} + ab$
- Left as an exercise to show that $(\mathbf{L} - a)(\mathbf{L} - b)T$ is the same as $(\mathbf{L}^2 - (a + b)\mathbf{L} + ab)T$

Lookup Table

- The annihilator $\mathbf{L} - a$ annihilates sequences of the form $\langle c_1 a^n \rangle$
- The annihilator $(\mathbf{L} - a)(\mathbf{L} - b)$ (where $a \neq b$) anihilates sequences of the form $\langle c_1 a^n + c_2 b^n \rangle$

Fibonnaci Sequence

- We now know enough to solve the Fibonnaci sequence
- Recall the Fibonnaci recurrence is $T(0) = 0$, $T(1) = 1$, and $T(n) = T(n-1) + T(n-2)$
- Let T_n be the n-th element in the sequence
- Then we've got:

$$T = \langle T_0, T_1, T_2, T_3, \cdots \rangle \quad (1)$$
$$\mathbf{L}T = \langle T_1, T_2, T_3, T_4, \cdots \rangle \quad (2)$$
$$\mathbf{L}^2 T = \langle T_2, T_3, T_4, T_5, \cdots \rangle \quad (3)$$

- Thus $\mathbf{L}^2 T - \mathbf{L}T - T = \langle 0, 0, 0, \cdots \rangle$
- In other words, $\mathbf{L}^2 - \mathbf{L} - 1$ is an annihilator for T

Factoring

- $\mathbf{L}^2 - \mathbf{L} - 1$ is an annihilator that is not in our lookup table
- However, we can *factor* this annihilator (using the quadratic formula) to get something similar to what's in the lookup table
- $\mathbf{L}^2 - \mathbf{L} - 1 = (\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$.

Quadratic Formula

"Me fail English? That's Unpossible!" - Ralph, the Simpsons

High School Algebra Review:

- To factor something of the form $ax^2 + bx + c$, we use the *Quadratic Formula*:
- $ax^2 + bx + c$ factors into $(x - \phi)(x - \hat{\phi})$, where:

$$\phi = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (4)$$

$$\hat{\phi} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (5)$$

Example

- To factor: $\mathbf{L}^2 - \mathbf{L} - 1$
- Rewrite: $1 * \mathbf{L}^2 - 1 * \mathbf{L} - 1$, $a = 1$, $b = -1$, $c = -1$
- From Quadratic Formula: $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$
- So $\mathbf{L}^2 - \mathbf{L} - 1$ factors to $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})$

Back to Fibonnaci

- Recall the Fibonnaci recurrence is $T(0) = 0$, $T(1) = 1$, and $T(n) = T(n-1) + T(n-2)$
- We've shown the annihilator for T is $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$
- If we look this up in the "Lookup Table", we see that the sequence T must be of the form $\langle c_1\phi^n + c_2\hat{\phi}^n\rangle$
- All we have left to do is solve for the constants c_1 and c_2
- Can use the base cases to solve for these

Finding the Constants

- We know $T = \langle c_1\phi^n + c_2\hat{\phi}^n\rangle$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$
- We know

$$\begin{aligned} T(0) &= c_1 + c_2 = 0 && (6) \\ T(1) &= c_1\phi + c_2\hat{\phi} = 1 && (7) \end{aligned}$$

- We've got two equations and two unknowns
- Can solve to get $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$,

The Punchline

- Recall Fibonnaci recurrence: $T(0) = 0$, $T(1) = 1$, and $T(n) = T(n-1) + T(n-2)$
- The final explicit formula for $T(n)$ is thus:

$$T(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

(Amazingly, $T(n)$ is *always* an integer, in spite of all of the square roots in its formula.)

A Problem

- Our lookup table has a big gap: What does $(\mathbf{L}-a)(\mathbf{L}-a)$ annihilate?
- It turns out it annihilates sequences such as $\langle na^n\rangle$

Example

$$\begin{aligned} (\mathbf{L}-a)\langle na^n\rangle &= \langle (n+1)a^{n+1} - (a)na^n\rangle \\ &= \langle (n+1)a^{n+1} - na^{n+1}\rangle \\ &= \langle (n+1-n)a^{n+1}\rangle \\ &= \langle a^{n+1}\rangle \\ (\mathbf{L}-a)^2\langle na^n\rangle &= (\mathbf{L}-a)\langle a^{n+1}\rangle \\ &= \langle 0\rangle \end{aligned}$$

Generalization

- It turns out that $(\mathbf{L}-a)^d$ annihilates sequences of the form $\langle p(n)a^n \rangle$ where $p(n)$ is any polynomial of degree $d-1$
- Example: $(\mathbf{L}-1)^3$ annihilates the sequence $\langle n^2 * 1^n \rangle = \langle 1, 4, 9, 16, 25 \rangle$ since $p(n) = n^2$ is a polynomial of degree $d-1=2$

Lookup Table

- $(\mathbf{L}-a)$ annihilates only all sequences of the form $\langle c_0 a^n \rangle$
- $(\mathbf{L}-a)(\mathbf{L}-b)$ annihilates only all sequences of the form $\langle c_0 a^n + c_1 b^n \rangle$
- $(\mathbf{L}-a_0)(\mathbf{L}-a_1)\ldots(\mathbf{L}-a_k)$ annihilates only sequences of the form $\langle c_0 a_0^n + c_1 a_1^n + \ldots c_k a_k^n \rangle$, here $a_i \neq a_j$, when $i \neq j$
- $(\mathbf{L}-a)^2$ annihilates only sequences of the form $\langle (c_0 n + c_1) a^n \rangle$
- $(\mathbf{L}-a)^k$ annihilates only sequences of the form $\langle p(n) a^n \rangle$, $degree(p(n)) = k-1$

Lookup Table

$$(\mathbf{L}-a_0)^{b_0}(\mathbf{L}-a_1)^{b_1}\ldots(\mathbf{L}-a_k)^{b_k}$$

annihilates only sequences of the form:

$$\langle p_1(n)a_0^n + p_2(n)a_1^n + \ldots p_k(n)a_k^n \rangle$$

where $p_i(n)$ is a polynomial of degree $b_i - 1$ (and $a_i \neq a_j$, when $i \neq j$)

Examples

- Q: What does $(\mathbf{L}-3)(\mathbf{L}-2)(\mathbf{L}-1)$ annihilate?
- A: $c_0 1^n + c_1 2^n + c_2 3^n$
- Q: What does $(\mathbf{L}-3)^2(\mathbf{L}-2)(\mathbf{L}-1)$ annihilate?
- A: $c_0 1^n + c_1 2^n + (c_2 n + c_3) 3^n$
- Q: What does $(\mathbf{L}-1)^4$ annihilate?
- A: $(c_0 n^3 + c_1 n^2 + c_2 n + c_3) 1^n$
- Q: What does $(\mathbf{L}-1)^3(\mathbf{L}-2)^2$ annihilate?
- A: $(c_0 n^2 + c_1 n + c_2) 1^n + (c_3 n + c_4) 2^n$

Annihilator Method

- Write down the annihilator for the recurrence
- Factor the annihilator
- Look up the factored annihilator in the "Lookup Table" to get general solution
- Solve for constants of the general solution by using initial conditions

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Lookup Table

$$(\mathbf{L}-a_0)^{b_0}(\mathbf{L}-a_1)^{b_1}\ldots(\mathbf{L}-a_k)^{b_k}$$

annihilates only sequences of the form:

$$\langle p_0(n)a_0^n + p_1(n)a_1^n + \ldots p_k(n)a_k^n\rangle$$

where $p_i(n)$ is a polynomial of degree $b_i - 1$ (and $a_i \neq a_j$, when $i \neq j$)

Examples

- Q: What does $(\mathbf{L}-3)(\mathbf{L}-2)(\mathbf{L}-1)$ annihilate?
- A: $c_0 1^n + c_1 2^n + c_2 3^n$
- Q: What does $(\mathbf{L}-3)^2(\mathbf{L}-2)(\mathbf{L}-1)$ annihilate?
- A: $c_0 1^n + c_1 2^n + (c_2 n + c_3)3^n$
- Q: What does $(\mathbf{L}-1)^4$ annihilate?
- A: $(c_0 n^3 + c_1 n^2 + c_2 n + c_3)1^n$
- Q: What does $(\mathbf{L}-1)^3(\mathbf{L}-2)^2$ annihilate?
- A: $(c_0 n^2 + c_1 n + c_2)1^n + (c_3 n + c_4)2^n$

Example

Consider the recurrence $T(n) = 7T(n-1) - 16T(n-2) + 12T(n-3)$, $T(0) = 1$, $T(1) = 5$, $T(2) = 17$

- **Write down the annihilator**: From the definition of the sequence, we can see that $\mathbf{L}^3T - 7\mathbf{L}^2T + 16\mathbf{L}T - 12T = 0$, so the annihilator is $\mathbf{L}^3 - 7\mathbf{L}^2 + 16\mathbf{L} - 12$
- **Factor the annihilator**: We can factor by hand or using a computer program to get $\mathbf{L}^3 - 7\mathbf{L}^2 + 16\mathbf{L} - 12 = (\mathbf{L}-2)^2(\mathbf{L}-3)$
- **Look up to get general solution**: The annihilator $(\mathbf{L}-2)^2(\mathbf{L}-3)$ annihilates sequences of the form $\langle (c_0 n + c_1)2^n + c_2 3^n \rangle$
- **Solve for constants**: $T(0) = 1 = c_1 + c_2$, $T(1) = 5 = 2c_0 + 2c_1 + 3c_2$, $T(2) = 17 = 8c_0 + 4c_1 + 9c_2$. We've got three equations and three unknowns. Solving by hand, we get that $c_0 = 1, c_1 = 0, c_2 = 1$. **Thus:** $T(n) = n2^n + 3^n$

Example (II)

Consider the recurrence $T(n) = 2T(n-1) - T(n-2)$, $T(0) = 0$, $T(1) = 1$

- **Write down the annihilator**: From the definition of the sequence, we can see that $\mathbf{L}^2T - 2\mathbf{L}T + T = 0$, so the annihilator is $\mathbf{L}^2 - 2\mathbf{L} + 1$
- **Factor the annihilator**: We can factor by hand or using the quadratic formula to get $\mathbf{L}^2 - 2\mathbf{L} + 1 = (\mathbf{L}-1)^2$
- **Look up to get general solution**: The annihilator $(\mathbf{L}-1)^2$ annihilates sequences of the form $(c_0 n + c_1)1^n$
- **Solve for constants**: $T(0) = 0 = c_1$, $T(1) = 1 = c_0 + c_1$, We've got two equations and two unknowns. Solving by hand, we get that $c_0 = 0, c_1 = 1$. **Thus:** $T(n) = n$

At Home Exercise

Consider the recurrence $T(n) = 6T(n-1) - 9T(n-2)$, $T(0) = 1$, $T(1) = 6$

- Q1: What is the annihilator of this sequence?
- Q2: What is the factored version of the annihilator?
- Q3: What is the general solution for the recurrence?
- Q4: What are the constants in this general solution?

(Note: You can check that your general solution works for $T(2)$)

Non-homogeneous terms

- Consider a recurrence of the form $T(n) = T(n-1) + T(n-2) + k$ where k is some constant
- The terms in the equation involving T (i.e. $T(n-1)$ and $T(n-2)$) are called the *homogeneous* terms
- The other terms (i.e.k) are called the *non-homogeneous* terms

Example

- In a *height-balanced tree*, the height of two subtrees of any node differ by at most one
- Let $T(n)$ be the smallest number of nodes needed to obtain a height balanced binary tree of height n
- Q: What is a recurrence for $T(n)$?
- A: Divide this into smaller subproblems
  - To get a height-balanced tree of height n with the smallest number of nodes, need one subtree of height $n-1$, and one of height $n-2$, plus a root node
  - Thus $T(n) = T(n-1) + T(n-2) + 1$

Example

- Let's solve this recurrence: $T(n) = T(n-1) + T(n-2) + 1$ (Let $T_n = T(n)$, and $T = \langle T_n \rangle$)
- We know that $(\mathbf{L}^2 - \mathbf{L} - 1)$ annihilates the homogeneous terms
- Let's apply it to the entire equation:

$$\begin{aligned}(\mathbf{L}^2 - \mathbf{L} - 1)\langle T_n \rangle &= \mathbf{L}^2\langle T_n \rangle - \mathbf{L}\langle T_n \rangle - 1\langle T_n \rangle \\ &= \langle T_{n+2} \rangle - \langle T_{n+1} \rangle - \langle T_n \rangle \\ &= \langle T_{n+2} - T_{n+1} - T_n \rangle \\ &= \langle 1, 1, 1, \cdots \rangle\end{aligned}$$

Example

- This is close to what we want but we still need to annihilate $\langle 1, 1, 1, \cdots \rangle$
- It's easy to see that $\mathbf{L} - 1$ annihilates $\langle 1, 1, 1, \cdots \rangle$
- Thus $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)$ annihilates $T(n) = T(n-1) + T(n-2) + 1$
- When we factor, we get $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})(\mathbf{L} - 1)$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$.

Lookup

- Looking up $(\mathbf{L} - \phi)(\mathbf{L} - \hat{\phi})(\mathbf{L} - 1)$ in the table
- We get $T(n) = c_1\phi^n + c_2\hat{\phi}^n + c_3 1^n$
- If we plug in the appropriate initial conditions, we can solve for these three constants
- We'll need to get equations for $T(2)$ in addition to $T(0)$ and $T(1)$

General Rule

To find the annihilator for recurrences with non-homogeneous terms, do the following:

- Find the annihilator a_1 for the homogeneous part
- Find the annihilator a_2 for the non-homogeneous part
- The annihilator for the whole recurrence is then $a_1 a_2$

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + 2$.
- The residue is $\langle 2, 2, 2, \cdots \rangle$ and
- The annihilator is still $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)$, but the equation for $T(2)$ changes!

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + 2^n$.
- The residue is $\langle 1, 2, 4, 8, \cdots \rangle$ and
- The annihilator is now $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 2)$.

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + n$.
- The residue is $\langle 1, 2, 3, 4, \cdots \rangle$
- The annihilator is now $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)^2$.

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + n^2$.
- The residue is $\langle 1, 4, 9, 16, \cdots \rangle$ and
- The annihilator is $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)^3$.

Another Example

- Consider $T(n) = T(n-1) + T(n-2) + n^2 - 2^n$.
- The residue is $\langle 1-1, 4-4, 9-8, 16-16, \cdots \rangle$ and the
- The annihilator is $(\mathbf{L}^2 - \mathbf{L} - 1)(\mathbf{L} - 1)^3(\mathbf{L} - 2)$.

In Class Exercise

- Consider $T(n) = 3 * T(n-1) + 3^n$
- Q1: What is the homogeneous part, and what annihilates it?
- Q2: What is the non-homogeneous part, and what annihilates it?
- Q3: What is the annihilator of $T(n)$, and what is the general form of the recurrence?