ON THE EQUATIONAL DEFINABILITY OF BROUWER-ZADEH LATTICES

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Abstract. We give an axiomatisation of the variety of Brouwer-Zadeh lattices, suitable for applications to quantum theory.

1. Introduction

A *Brouwer-Zadeh poset* (resp. *lattice*) is a structure $\langle A; \leq, \neg, \sim, 0, 1\rangle$ of type $\langle 2, 1, 1, 0, 0\rangle$ where $\langle A; \leq, 0, 1\rangle$ is a bounded poset (resp. bounded lattice), such that the following universal sentences are satisfied:

$$\neg\neg x \approx x \tag{1}$$

$$x \leq y \supset \neg y \leq \neg x \tag{2}$$

$$x \wedge \neg x \leq y \vee \neg y \tag{3}$$

$$x \leq {\sim}{\sim} x \tag{4}$$

$$x \leq y \supset {\sim} y \leq {\sim} x \tag{5}$$

$$x \wedge {\sim} x \approx \mathbf{0} \tag{6}$$

$$\neg{\sim} x \approx {\sim}{\sim} x. \tag{7}$$

(Here for readability we are omitting all quantification of variables.)

Brouwer-Zadeh posets (resp. lattices) were introduced by Cattaneo and Marino [2] in connection with unsharp approaches to quantum theory. Given a Hilbert space $\mathcal{H}$, the set $E(\mathcal{H})$ of all effects can be naturally structured as a Brouwer-Zadeh poset [3]. Moreover, every Brouwer-Zadeh poset embeds as a Brouwer-Zadeh poset into a complete Brouwer-Zadeh lattice, namely its MacNeille completion [5]. Brouwer-Zadeh lattices and their corresponding logics have since been studied by a variety of authors, including Cattaneo and Nisticó [3], Giuntini [6, 7] and Cattaneo and Ciucci [1]. For a survey of quantum logics and

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their associated algebraic semantics, including Brouwer-Zadeh lattices, see Dalla Chiara and Giuntini [4].

When the partial ordering of a Brouwer-Zadeh poset $\mathbf{A}$ is a lattice ordering, no harm comes from systematically confusing $\mathbf{A}$ with its underlying *algebra* $\langle A; \wedge, \vee, \neg, \sim, 0, 1\rangle$. (Here $\langle A; \wedge, \vee\rangle$ is the lattice reduct of $\mathbf{A}$.) In consequence, we may view a Brouwer-Zadeh lattice as an algebra $\langle A; \wedge, \vee, \neg, \sim, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ satisfying the identities and quasi-identities (1)–(7) together with the lattice identities:

$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z \tag{8}$$

$$x \vee (y \vee z) \approx (x \vee y) \vee z \tag{9}$$

$$x \wedge x \approx x \tag{10}$$

$$x \vee x \approx x \tag{11}$$

$$x \wedge y \approx y \wedge x \tag{12}$$

$$x \vee y \approx y \vee x \tag{13}$$

$$x \wedge (x \vee y) \approx x \tag{14}$$

$$x \vee (x \wedge y) \approx x \tag{15}$$

$$x \wedge \mathbf{0} \approx \mathbf{0} \tag{16}$$

$$x \vee \mathbf{1} \approx \mathbf{1}. \tag{17}$$

(The Brouwer-Zadeh lattice ordering $\leq$ is recovered in the usual way upon setting $x \leq y$ if and only if $x \wedge y \approx x$.) The class BZ of all Brouwer-Zadeh lattices is thus a quasivariety. In fact, an elementary lattice-theoretic argument establishes that an algebra $\langle A; \wedge, \vee, \neg, \sim, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ is a Brouwer-Zadeh lattice if and only if it satisfies the identities (1), (3)–(4) and (6)–(17), together with the identities $\neg(x \vee y) \wedge (\neg x \wedge \neg y) \approx \neg(x \vee y)$ and $\sim(x \vee y) \wedge (\sim x \wedge \sim y) \approx \sim(x \vee y)$. Hence BZ is actually equationally definable.

In this note, we present an alternative axiomatisation of the variety of Brouwer-Zadeh lattices, more suited to applications in quantum theory than the equational basis described above. In particular, we establish the following

Theorem. *An algebra* $\langle A; \wedge, \vee, \neg, \sim, 0, 1\rangle$ *of type* $\langle 2, 2, 1, 1, 0, 0\rangle$ *is a Brouwer-Zadeh lattice if and only if it satisfies the identities* (1)*,* (3)*–*(4) *and* (6)*–*(17)*, together with the identities*

$$\neg(x \vee y) \approx \neg x \wedge \neg y \tag{18}$$

$$\sim(x \vee y) \approx \sim x \wedge \sim y. \tag{19}$$

Hence the class of all Brouwer-Zadeh lattices is a variety.

2. The Proof

Let **A** be an algebra $\langle A; \wedge, \vee, \neg, \sim, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ satisfying the identities (1), (3)–(4), (6)–(17) and (18)–(19). Suppose $a \le b$. Then $b \wedge a = a$, so $b = b \vee (b \wedge a) = b \vee a$, whence $\neg b = \neg(b \vee a) = \neg b \wedge \neg a$ by (18). Thus $\neg b \le \neg a$. An analogous argument using (19) establishes $\sim b \le \sim a$. Hence **A** is a Brouwer-Zadeh lattice.

In the (machine-oriented) proof of the converse given below, the justification $[i \to j]$ indicates paramodulation from i into j; that is, unifying the left-hand side of i with a subterm of j, instantiating j with the corresponding substitution, and replacing the subterm with the corresponding instance of the right-hand side of i. Also, the justification $[i, j]$ indicates an application of modus ponens with major premiss i and minor premiss j. The two identities to be established are derived at steps 77 and 94 respectively. These steps of the proof are flagged with '*' for easy identification.

1. $\neg(\neg x) \approx x$ [(1)]
2. $x \wedge y \approx x \;\supset\; \neg y \wedge \neg x \approx \neg y$ [(2)]
3. $x \wedge \sim(\sim x) \approx x$ [(4)]
4. $x \wedge y \approx x \;\supset\; \sim y \wedge \sim x \approx \sim y$ [(5)]
5. $\neg(\sim x) \approx \sim(\sim x)$ [(7)]
6. $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$ [(8)]
7. $x \vee (y \vee z) \approx (x \vee y) \vee z$ [(9)]
8. $x \wedge x \approx x$ [(10)]
9. $x \wedge y \approx y \wedge x$ [(12)]
10. $x \vee y \approx y \vee x$ [(13)]
11. $x \wedge (x \vee y) \approx x$ [(14)]
12. $x \vee (x \wedge y) \approx x$ [(15)]
13. $\sim(\neg(\sim x)) \approx \neg(\sim(\sim x))$ [5 → 5]
14. $\neg(\sim(\sim x)) \approx \sim x$ [5 → 1]
15. $x \wedge \neg(\sim x) \approx x$ [5 → 3]
16. $(x \wedge y) \wedge y \approx x \wedge y$ [8 → 6]
17. $\neg(\sim x) \vee y \approx y \vee \sim(\sim x)$ [5 → 10]
18. $(x \vee y) \vee z \approx (z \vee x) \vee y$ [7 → 10]
19. $\sim(x \vee y) \wedge \sim x \approx \sim(x \vee y)$ [4, 11]
20. $\neg(x \vee y) \wedge \neg x \approx \neg(x \vee y)$ [2, 11]
21. $x \wedge (y \vee x) \approx x$ [10 → 11]
22. $x \vee (y \wedge x) \approx x$ [9 → 12]
23. $(x \wedge y) \vee (x \wedge (y \wedge z)) \approx x \wedge y$ [6 → 12]
24. $x \wedge (y \wedge \neg(\sim(x \wedge y))) \approx x \wedge y$ [6 → 15]

25. $\sim(x \vee y) \wedge \sim y \approx \sim(x \vee y)$ [4, 21]
26. $\neg(x \vee y) \wedge \neg y \approx \neg(x \vee y)$ [2, 21]
27. $\neg(\sim x) \vee x \approx \neg(\sim x)$ [15 → 22]
28. $\sim(\neg(\sim x)) \approx \sim x$ [14 → 13]
29. $\sim x \wedge \sim(y \wedge x) \approx \sim x$ [4, 16]
30. $\neg x \wedge \neg(y \wedge x) \approx \neg x$ [2, 16]
31. $x \vee \neg(\sim x) \approx \neg(\sim x)$ [10 → 27]
32. $\sim x \wedge \sim(y \wedge \neg(\sim x)) \approx \sim(\neg(\sim x))$ [28 → 29]
33. $\sim x \wedge \sim(x \wedge y) \approx \sim x$ [9 → 29]
34. $\sim(x \wedge y) \vee \sim y \approx \sim(x \wedge y)$ [29 → 22]
35. $\neg x \wedge \neg(x \wedge y) \approx \neg x$ [9 → 30]
36. $\neg(x \wedge y) \vee \neg y \approx \neg(x \wedge y)$ [30 → 22]
37. $\sim(x \wedge y) \vee \sim x \approx \sim(x \wedge y)$ [33 → 22]
38. $\neg(x \wedge y) \vee \neg x \approx \neg(x \wedge y)$ [35 → 22]
39. $\sim(x \vee y) \wedge \sim y \approx \sim(y \vee x)$ [10 → 19]
40. $\neg x \wedge \neg(x \vee y) \approx \neg(x \vee y)$ [9 → 20]
41. $\neg x \wedge \neg(y \vee x) \approx \neg(y \vee x)$ [9 → 26]
42. $\sim x \vee \sim(y \wedge x) \approx \sim(y \wedge x)$ [10 → 34]
43. $\neg x \vee \neg(y \wedge x) \approx \neg(y \wedge x)$ [10 → 36]
44. $\sim(\sim x \wedge y) \vee \neg(\sim x) \approx \sim(\sim x \wedge y)$ [5 → 37]
45. $\neg x \vee \neg(x \wedge y) \approx \neg(x \wedge y)$ [10 → 38]
46. $\sim(x \vee y) \approx \sim(y \vee x)$ [25 → 39]
47. $\sim x \wedge \sim(y \vee x) \approx \sim(x \vee y)$ [9 → 39]
48. $\sim(\neg(\sim x) \vee y) \approx \sim(y \vee \sim(\sim x))$ [5 → 46]
49. $x \wedge \neg(\neg x \vee y) \approx \neg(\neg x \vee y)$ [1 → 40]
50. $\neg x \wedge \neg(\neg(\sim x)) \approx \neg(x \vee \neg(\sim x))$ [31 → 40]
51. $x \wedge \neg(y \vee \neg x) \approx \neg(y \vee \neg x)$ [1 → 41]
52. $(x \wedge \sim y) \vee (x \wedge \sim(y \vee z)) \approx x \wedge \sim y$ [47 → 23]
53. $\sim(\sim(\sim x) \vee \sim(\sim y)) \approx \sim(\neg(\sim y) \vee \neg(\sim x))$ [5 → 48]
54. $\sim x \wedge \neg(\neg(\sim x) \vee y) \approx \neg(\sim(\sim x) \vee y)$ [5 → 49]
55. $(x \wedge y) \vee (x \wedge \neg(\neg y \vee z)) \approx x \wedge y$ [49 → 23]
56. $x \vee \neg(\neg x \vee y) \approx x$ [49 → 12]
57. $x \vee \neg(y \vee \neg x) \approx x$ [10 → 56]
58. $\sim x \wedge \sim(y \wedge \neg(\sim x)) \approx \sim x$ [28 → 32]
59. $\sim x \wedge \sim(\neg(\sim x \vee y)) \approx \sim x$ [20 → 58]
60. $\neg(x \vee \neg(\sim x)) \approx \neg x \wedge \sim x$ [1 → 50]
61. $\neg x \wedge \sim x \approx \neg(\neg(\sim x))$ [31 → 60]
62. $\neg x \wedge \sim x \approx \sim x$ [1 → 61]
63. $(x \wedge y) \vee \neg(\neg y \vee \neg x) \approx x \wedge y$ [51 → 55]
64. $(\neg x \vee \neg y) \vee \neg(y \wedge x) \approx \neg x \vee \neg y$ [63 → 57]
65. $(x \wedge \sim(y \vee z)) \vee (x \wedge \sim y) \approx x \wedge \sim y$ [10 → 52]
66. $\neg(\sim(\sim x) \vee y) \approx \neg(\neg(\sim x) \vee y)$ [49 → 54]

67. $\neg(\sim(\sim x) \vee y) \approx \neg(y \vee \neg(\sim x))$ [10 → 66]
68. $\neg x \vee (\neg y \vee \neg(y \wedge x)) \approx \neg x \vee \neg y$ [7 → 64]
69. $\sim(x \vee y) \vee (\neg(x \vee y) \wedge \sim x) \approx \neg(x \vee y) \wedge \sim x$ [62 → 65]
70. $\neg x \vee \neg(y \wedge x) \approx \neg x \vee \neg y$ [45 → 68]
71. $\neg x \vee \neg y \approx \neg(y \wedge x)$ [43 → 70]
72. $\neg(x \wedge \neg y) \approx y \vee \neg x$ [1 → 71]
73. $\neg(\neg x \wedge y) \approx \neg y \vee x$ [1 → 71]
74. $\neg(x \vee \neg y) \approx y \wedge \neg x$ [72 → 1]
75. $\neg(\sim(\sim x) \vee y) \approx \sim x \wedge \neg y$ [67 → 74]
76. $\neg(x \vee \neg y) \approx \neg x \wedge y$ [9 → 74]
*77. $\neg x \wedge \neg y \approx \neg(x \vee y)$ [1 → 76]
78. $\sim x \wedge \neg(\sim(y \wedge \sim x)) \approx \neg(\sim(y \wedge \sim x))$ [42 → 75]
79. $x \wedge \neg(\sim(x \wedge \sim y)) \approx x \wedge \sim y$ [78 → 24]
80. $\sim(x \wedge \sim y) \vee \neg x \approx \neg(x \wedge \sim y)$ [79 → 72]
81. $\sim(\sim x \wedge \sim y) \approx \neg(\sim x \wedge \sim y)$ [44 → 80]
82. $\sim(\neg(\sim x \wedge \sim y)) \approx \neg(\sim(\sim x \wedge \sim y))$ [81 → 5]
83. $\neg(\sim(\sim x \wedge \sim y)) \approx \sim x \wedge \sim y$ [81 → 1]
84. $\sim(\neg(\sim x \wedge \sim y)) \approx \sim x \wedge \sim y$ [83 → 82]
85. $\sim(\sim(\sim x) \vee \sim(\sim y)) \approx \sim(\neg(\sim x \wedge \sim y))$ [71 → 53]
86. $\sim(\sim(\sim x) \vee \sim(\sim y)) \approx \sim x \wedge \sim y$ [84 → 85]
87. $\sim(x \vee y) \wedge \sim(\neg(\neg(x \vee y) \wedge \sim x)) \approx \sim(x \vee y)$ [69 → 59]
88. $\sim(x \vee y) \wedge \sim(\neg(\sim x) \vee (x \vee y)) \approx \sim(x \vee y)$ [73 → 87]
89. $\sim((x \vee y) \vee \neg(\sim x)) \approx \sim(x \vee y)$ [47 → 88]
90. $\sim((\neg(\sim x) \vee x) \vee y) \approx \sim(x \vee y)$ [18 → 89]
91. $\sim(\neg(\sim x) \vee y) \approx \sim(x \vee y)$ [27 → 90]
92. $\sim(x \vee \sim(\sim y)) \approx \sim(y \vee x)$ [17 → 91]
93. $\sim(x \vee \sim(\sim y)) \approx \sim y \wedge \sim x$ [86 → 92]
*94. $\sim x \wedge \sim y \approx \sim(x \vee y)$ [92 → 93]

The preceding proof was obtained with the assistance of the automated reasoning program Otter [8], using the method of proof sketches [10]. For examples of the application of automated reasoning to a wide range of problems in equational logic, see [9].

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