The Moufang Laws, Global and Local

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Global:

Commutative: $xy = yx$

Associative: $xy \cdot z = x \cdot yz$

Local:

Commutant: $\mathsf{C}(L) = \{a : ax = xa\}$

Left Nucleus: $\mathsf{N}_\lambda(L) = \{a : a \cdot xy = ax \cdot y\}$
Middle Nucleus: $\mathsf{N}_\mu(L) = \{a : x \cdot ay = xa \cdot y\}$
Right Nucleus: $\mathsf{N}_\rho(L) = \{a : x \cdot ya = x \cdot ya\}$

Nucleus: $\mathsf{N}(L) = \mathsf{N}_\lambda(L) \cap \mathsf{N}_\mu(L) \cap \mathsf{N}_\rho(L)$

Commutants need not be subloops (some Bol loops, for instance). But in some varieties, e.g., groups, Moufang loops, they are.

All four nuclei are subloops.

Global

A: $z(xy \cdot z) = zx \cdot yz$

B: $(z \cdot xy)z = zx \cdot yz$

C: $z(x \cdot zy) = (zx \cdot z)y$

D: $(xz \cdot y)z = x(z \cdot yz)$

In loops, each of these four is equivalent to the other three. In fact, the same is true in quasigroups (and in this case, they're loops [Kunen '96]).

Local

A2: $a(xy \cdot a) = ax \cdot ya$

A1x: $z(ay \cdot z) = za \cdot yz$

A1y: $z(xa \cdot z) = zx \cdot az$

B2: $(a \cdot xy)a = ax \cdot ya$

B1x: $(z \cdot ay)z = za \cdot yz$

B1y: $(z \cdot xa)z = zx \cdot az$

C2: $a(x \cdot ay) = (ax \cdot a)y$

C1x: $z(a \cdot zy) = (za \cdot z)y$

C1y: $z(x \cdot za) = (zx \cdot z)a$

D2: $(xa \cdot y)a = x(a \cdot ya)$

D1x: $(az \cdot y)z = a(z \cdot yz)$

D1y: $(xz \cdot a)z = x(z \cdot az)$

In loops, A2 and B2 are equivalent. The are no other implications. Hence, there are 11 possible definitions of “Moufang element”.

Traditionally, A2 is taken to be the definition of Moufang element. Explicitly, an element a in a loop L is called a *Moufang element* if it satisfies $a(xy \cdot a) = ax \cdot ya$. Why is A2 (B2) so privileged?

In a left (or right) inverse property loop, the set of Moufang elements forms a subloop [Florja, ’65].

The set of Moufang elements in an arbitrary loop, though, need not be a subloop.

0	1	2	3	4	5	6	7	8	9	10	11
1	0	3	2	5	4	7	6	10	11	8	9
2	4	0	5	1	3	8	9	6	7	11	10
3	5	1	4	0	2	9	8	11	10	6	7
4	2	5	0	3	1	10	11	7	6	9	8
5	3	4	1	2	0	11	10	9	8	7	6
6	7	8	10	9	11	0	1	2	3	4	5
7	6	9	11	8	10	1	0	3	2	5	4
8	10	6	7	11	9	2	4	0	5	1	3
9	11	7	6	10	8	4	2	5	0	3	1
10	8	11	9	6	7	3	5	1	4	0	2
11	9	10	8	7	6	5	3	4	1	2	0

1 and 2 are Moufang elements, but $1 \cdot 2$ is not. This loop is flexible and solvable; it satisfies the AAIP (but perforce, neither LIP or RIP); its three nuclei, as well as its commutant, are all trivial; and it's only two nontrivial associators are $3 = 1 \cdot 2$ and $4 = 2 \cdot 1$. This construction generalizes to a loop of order $4n, n \geq 3$ with exactly n associators. I know of two other infinite families.

Translations: $yL_x = xy = xR_y$

Autotopisms: $xG \cdot yH = (x \cdot y)K$, denoted by (G, H, K)

Four autotopisms in Moufang loops, corresponding to (A), (B), (C), and (D):

A: $z(xy \cdot z) = zx \cdot yz : (L_z, R_z, R_zL_z)$
B: $(z \cdot xy)z = zx \cdot yz : (L_z, R_z, L_zR_z)$
C: $z(x \cdot zy) = (zx \cdot z)y : (L_zR_z, L_z^{-1}, L_z)$
D: $(xz \cdot y)z = x(z \cdot yz) : (R_z^{-1}, R_zL_z, R_z)$

"Elementwise", we get:

A2: $a(xy \cdot a) = ax \cdot ya : (L_a, R_a, R_aL_a)$
B2: $(a \cdot xy)a = ax \cdot ya : (L_a, R_a, L_aR_a)$
C2: $a(x \cdot ay) = (ax \cdot a)y : (L_aR_a, L_a^{-1}, L_a)$
D2: $(xa \cdot y)a = x(a \cdot ya) : (R_a^{-1}, R_aL_a, R_a)$

So we see that there are three (formally four) reasonable definitions of Moufang element (including the traditional one), at least insofar as they are expressible via an autotopism:

left Moufang: $\mathsf{M}_\lambda(L) = \{a(x \cdot ay) = (ax \cdot a)y\}$
middle Moufang: $\mathsf{M}_\mu(L) = \{a(xy \cdot a) = ax \cdot ya\}$
right Moufang: $\mathsf{M}_\rho(L) = \{(xa \cdot y)a = x(a \cdot ya)\}$

Note: none of these is necessarily a subloop.

Moufang element:

$$\mathsf{M}(L) = \mathsf{M}_\lambda(L) \cap \mathsf{M}_\mu(L) \cap \mathsf{M}_\rho(L)$$

Theorem: $\mathsf{M}(L)$ is a subloop [Phillips, '08].

Note: in fact, A2 + C2 imply D2.

Recall:

A: $z(xy \cdot z) = zx \cdot yz$

B: $(z \cdot xy)z = zx \cdot yz$

C: $z(x \cdot zy) = (zx \cdot z)y$

D: $(xz \cdot y)z = x(z \cdot yz)$

Recall Kunen's result: a quasigroup satisfying any one of (A), (B), (C), or (D), is a Moufang loop.

Generalized globally: a divisible groupoid satisfying any one of (A), (B), (C), or (D), is a Moufang loop [Phillips, '08].

Kunen's result recast: a quasigroup satisfying any one of (A), (B), (C), or (D), has a 2-sided identity element.

Generalized locally: Let L be a groupoid containing an element a that is:

(1) A2, A1x, and A1y and such that L_a bijects and R_a is either onto or 1-1. Then, L has a two-sided identity element.

(2) C2, C1x, and C1y and such that L_a and R_a are both onto and such that $L_a L_a = L_{a^2}$. Then L has a two-sided identity element.

(Mirror statements for B and D.)

Miscellania

A loop L containing an element a that satisfies all 12 “Moufang conditions” must be flexible, right alternative, and left alternative.

If an element a in a flexible, alternative loop is A2, A1x, and A1y, then it is also B2, B1x, B1y, C2, C1x, C1y, D2, D1x, D1y.

Minimal size of loop with the following set of “Moufang elements” not a subloop:

A2: 12
A1x: 10
A1y: 8

C2: 12
C1x: 12
C1y: 12

Odd order examples?

An Application

Inner mapping group generated by:

$T(x) = L(x)^{-1}R(x)$

$R(x, y) = R(x)R(y)R(xy)^{-1}$

$L(x, y) = L(x)L(y)L(yx)^{-1}$

A-loop: all inner mappings are automorphisms.

Important examples: groups, commutative Moufang loops.

Inverse property A-loops are Moufang [Kinyon, Kunen, Phillips, 2002]

We will call an element a an *A-element* if, for each x, the following are automorphisms: $L(x, a), L(a, x), R(a, x), R(x, a), T(a)$.

Theorem: If a is an A-element in an inverse property loop, then a is a Moufang element [Phillips, '08].